Arch. Hist. Exact Sci. 61 (2007) 67–81
Digital Object Identifier (DOI) 10.1007/s00407-006-0115-7

Huygens' Methods for Determining Optical Parameters in Birefringence

JED Z. BUCHWALD

Christiaan Huygens' construction for the birefringence of the crystal Iceland spar has long raised questions concerning the experimental and computational methods that he deployed along the way to his final results, which were eventually printed in the *Traité de la Lumière* of 1690. The documentary record as presented in part in the magisterial edition of his *Oeuvres Complètes* does enable the historian to reconstruct much of the development.[1] The editors however both reordered the manuscript and omitted pages of calculations. A proper evaluation must accordingly examine the original material, which is preserved in Codex 9 of the Huygens notebooks at the Bibliotheek of the Rijksuniversiteit Leiden. Several omitted pages of the Codex contain numerical data from which a great deal can be concluded that is pertinent for understanding just how Huygens mated mathematics with measurement. What follows presents a new reconstruction of his methods based on the published and unpublished material.

From June of 1671 through 1681 Huygens resided for the most part in Paris as a founding member of the Académie des Sciences and as a client of Colbert, remaining there even during Louis XIV's war on the Dutch Provinces in 1672. His first notes on double refraction date from 1672 or 1673 and concern his early understanding of the subject after having read Erasmus Bartholin's recent publication.[2] These notes contain inklings (perhaps as a result of contacts with Fr. Ignace Pardies[3]) of what would evolve sometime during the next five years into Huygens' fully-fledged wave understanding of light.[4] Furthermore he seems by this time to have perceived the problem posed for any such account by the optics of Iceland spar, namely that the extraordinary ray in the crystal seemed to be oblique rather than perpendicular to its front.

[1] *Oeuvres Complètes de Christiaan Huygens*, 1888–1950, Société Hollandaise des Sciences, La Haye, Martinus Nijhoff.

[2] Bartholini, E. (1669). *Experimenta Crystalli Islandici Disdiaclastici Quibus mira & insolita Refraction detegitur*, Hafniae, Danielis Paulli.

[3] Dijksterhuis, F. (2004). *Lenses and Waves. Christiaan Huygens and the Mathematical Science of Optics in the Seventeenth Century*, *Series Archimedes*, Dordrecht, Kluwer Academic Publishers, p. 110.

[4] The most insightful account of the 17th century background to wave optics remains A. E. Shapiro (1973), "Kinematic optics: a study of the wave theory of light in the seventeenth century." *Archive for History of Exact Sciences* 11: 134–266.

For present purposes the most revealing documents were written from 1677 through 1679, beginning with an explicitly-dated diagram that depicts the geometry which Huygens had by then developed for the extraordinary refraction, together with Latin wording on its several radii, sketches of stacked spheres, and bearing the heading 'EYRHKA 6 Aug. 1677. Causam mirae refractionis in Crystallo Islandica'.[5] There follow two undated Latin paragraphs, each of which specifies various parameters. The first of the two[6] contains calculated values for crystal angles based on Huygens' use of spherical trigonometry. These values are precisely the same as the ones that can be deduced from the August 1677 diagram. The second of the two[7] is accompanied by a labeled figure which clearly shows a ray that passes through the crystal without refraction,[8] though this material dates from two years later (but see note 23 below). Pages 440–441 of the *Oeuvres* (extracted from another notebook) contain remarks in French (accompanied by a diagram) which Huygens titled "Observation faite le 3 Juillet 1679 qui prouve manifestement que ce n'est pas le rayon parallele aux costez du crystal qui passe sans refraction comme j'avois creu jusqu'icy". Two separately-labeled sections are next. The first is titled by Huygens "A Paris 6 Aoust 1679. EYRHKA. La confirmation de ma theorie de la lumiere et des refractions"; it examines refractions with crystals cut at various angles through the ellipsoid.[9] The second, which is undated but follows immediately in the notebook, discusses the difference between an expanding ellipsoid and an expanding sphere.[10]

Although it's clear from the August 1677 'Eureka' page that Huygens had worked with Iceland spar, the extent of his experimentation remains uncertain.[11] By the summer of 1679, two years later, he had begun a series of careful measurements, in major part because Ole Rømer had queried Huygens' remarks on the subject, possibly during a meeting on July 1 at the Paris Académie.[12] The July 3 extract noted above contains what is almost certainly Huygens' reply to Rømer, based on experiments and backed further over the next month by his work with cut crystals.

---

[5] *Oeuvres,* vol. XIX, pp. 427–9; pg. 45 in Codex 9. All further references to the *Oevres* are to vol. XIX.

[6] *Oeuvres,* pg. 430; pg. 48 in Codex 9.

[7] *Oeuvres,* pg. 430, but pg. 90 *verso* in Codex 9.

[8] *Oeuvres,* pg. 431, fig. 155 at top.

[9] *Oeuvres* pp.441–3; Codex 9, pg. 98.

[10] *Oeuvres* pg. 443; Codex 9, pg. 99.

[11] Dijksterhuis (p. 207) argues that Huygens before 1679 "had never measured the unrefracted oblique ray" or indeed any ray except for the one perpendicularly incident. That, in fact, "he developed the technique to measure the refraction of a ray only in 1679". This last claim is unlikely since the general technique did not have to be developed by Huygens: it is entirely simple and is implicit in Bartholin's *Experimenta* (see his Experiment XVII), as Dijksterhuis himself notes. Moreover, he needed something like it to measure the deviation at normal incidence. There is also no reason to doubt Huygens' claim in the published *Traité* that he had discovered some of the simpler properties of the extraordinary refraction by observation quite early on, since he certainly did work with images in 1672/3 – which is how he had discovered what a century and a quarter later would be named polarization by Malus (*Oeuvres*, pg. 413). Finally, and of greatest importance, we will see that the 1677 document bears unmistakable traces of a distinct measurement, albeit probably not of a refraction angle.

[12] Dijksterhuis, pp. 205–7.

The large, central diagram on the 'EYRHKA' page from the summer of 1677 (Fig. 1) depicts a principal section[13] of Huygens' new surface for the extraordinary refraction, together with various numbers. The smaller diagram at upper left depicts rays incident on the crystal from both left and right, while the words to the top and right provide definitions, a relation that determines the extraordinary refraction, and explanations. Just below and to the center left of the main diagram two numbers are added to produce a third. The far left of the page contains an algebraic sketch written upside down.

To understand how Huygens obtained these numbers we must first consider a major consequence of his structure. The basic geometry replaces the sphere of ordinary refraction with an (oblate) ellipsoid of revolution, and the consequence in question enables a refraction to be computed directly from an incident ray that lies in a principal section or in a plane normal to one. The consequence appears explicitly in Fig. 1 in the words just above and to the right of the small diagram at the upper left.

In Fig. 2 (which exaggerates the ellipticity for clarity) the line LA represents an incoming wave front whose corresponding ray is RA, drawn from a circle of radius AH; the interface is gAH, and point V is on the perpendicular to the interface from R. Within the crystal, the section gPSH of the ellipsoidal wavefront (center A), whose semi-major and semi-minor axes are respectively AP, AS (with AS the axis of rotation), determines the refracted front μC, whose corresponding refracted ray is AC. The line Lμ, perpendicular to the incoming front and touching the interface, represents the distance light travels in air in the same time that the surface gPSH is formed from the end of the front that is incident at A; point C on the refracted ray is determined by a tangent drawn through μ to gPSH. From C draw CX to the interface along a parallel to the refraction AB of a ray that is incident along the perpendicular FA (whose angle of refraction we shall call the normal deviation). Then, Huygens wrote above the diagram, the distance AX will satisfy the following relation:

$$\frac{AV}{AX} = \frac{L\mu}{AH} \text{ which is the same as } AX = \frac{AH^2 \sin(\angle FAR)}{L\mu} \tag{1}$$

Given (1), the procedure for computing the angle of refraction ($\angle IAC$) runs as follows, though Huygens specified the method only in his published account[14] – it does not appear explicitly in the earlier manuscripts:

$$\begin{aligned} AJ &= \tfrac{AB\sqrt{AH^2-AX^2}}{AH} \\ aB &= (AB)(AX)/AJ \\ aI &= aB + AB\sin(\angle IAB) \\ AI &= AB\cos(\angle IAB) \\ \angle IAC &= \tan^{-1}\left(\tfrac{aI}{AI}\right) \end{aligned} \tag{2}$$

Since Huygens had no specific need to compute a refraction from an incidence in 1677 it's possible that he had not by then developed (2). If he had, then he would have been able from it to reverse the calculation, and so to find AX from a measured refraction angle and the ellipse parameters as

[13] One that cuts perpendicularly down through any face along a line that bisects an obtuse facet angle.

[14] Huygens, *Traité*, sec. 34.

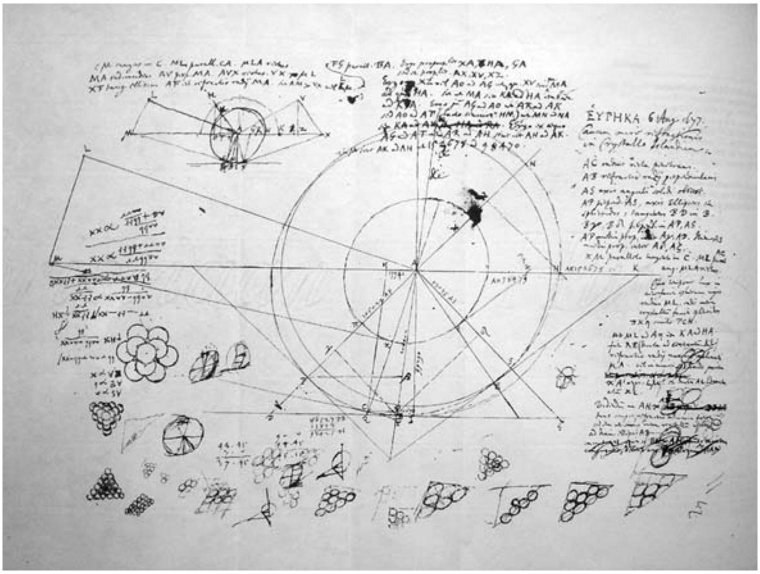

Fig. 1. The August 6, 1677 'EYRHKA' diagram (*Oeuvres*, insert before pg. 427; Codex 9, pg. 47)

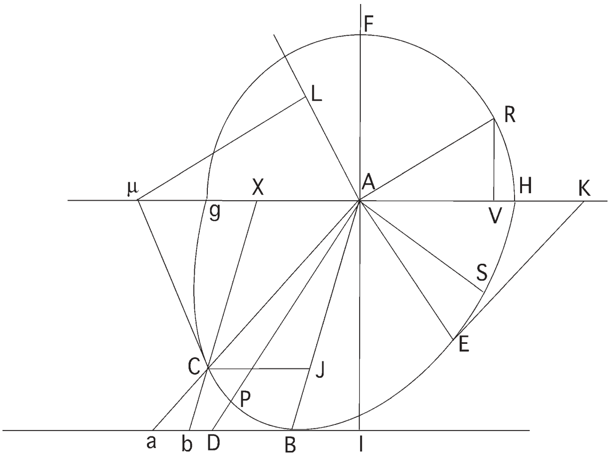

Fig. 2. Huygens' construction

$$AX = \frac{(AH)(AB)[\cos(\angle IAB)\tan(\angle IAC) - \sin(\angle IAB)]}{\sqrt{AB^2[\cos(\angle IAB)\tan(\angle IAC) - \sin(\angle IAB)]^2 + AH^2}} \tag{3}$$

Given AX from (3), then (1) will yield either the corresponding angle of incidence, given Lμ (the radius of the sphere in air), or else Lμ itself given the angle of incidence. Still, there is no evidence in the notebook that Huygens had (2), much less the ungainly expression (3) for AX.

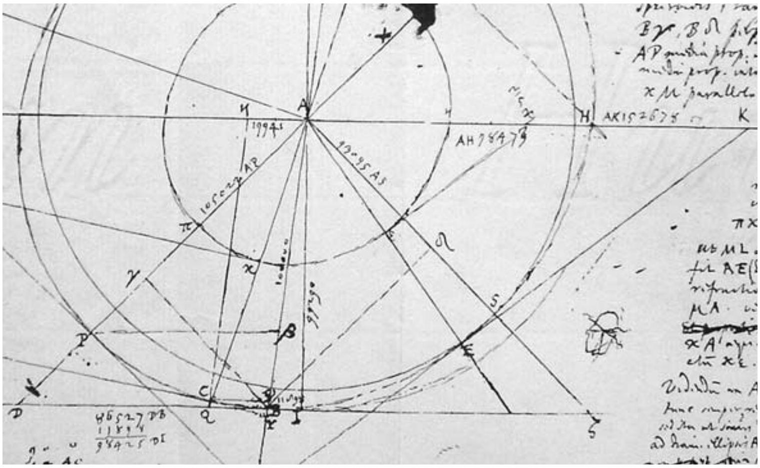

Fig. 3. Huygens' parameters in 1677

The value of AH (the section of the ellipse by the interface) as well as that of Lμ must be known to compute AX from (1). To find AH requires the orientation of the ellipsoid within the crystal as well as the angle of AB – the refraction of a normally-incident ray – whose magnitude is set to 100000 as a reference, thereby retaining six significant digits. Symmetry considerations dictate the ellipsoid's orientation in terms of the angle between crystal facets.[15] Only one other observation is then needed in order to determine Lμ, which must be found from some measurement in extraordinary refraction since it has to be specified in units of AB. This could be done by measuring both a refraction and its corresponding incidence; given both, (3) would yield AX and then L$\mu$from (1), but, again, there is no clear evidence that Huygens possessed (3). Note however (Fig. 3) that Huygens did depict a ray AC which goes straight through the crystal without deviation - the "radius recta penetrans", in his accompanying words. This undeviated ray intersects the tangent through B at point a (Fig. 2), and the corresponding value of AX is marked on the diagram. We will consider its significance below.

We can read many others of Huygens' values directly from the diagram and from the additional numbers below it and to the left (Table 1).[16] His figure and words together also specify the meaning of all of the marked points, excepting only one, namely I. However, the diagram places I visually along DB – the line through B and parallel to the interface. Moreover, in the summation to the left and below the figure Huygens has

[15] Since the refractions are the same in all three principal sections of the facets that meet at a solid obtuse angle the axis of revolution of the ellipsoid (AS) must lie in all three sections.

[16] Huygens' figure additionally depicts lines AE and EK. He describes AE as the refraction of a ray (from the left) that just scrapes the interface, while EK is the tangent to the spheroid at its intersection with AE. Considering the same (scraping) incidence of a ray AR from the right, we have tan($\angle$FAR) of incidence equal to Lμ/LA. As the incidence approaches 90° the interval LA must go to zero since Lμ is fixed, in which case Aμ becomes equal to Lμ (Fig. 2), and so (for scraping incidence from the left) Huygens' AK must also represent Lμ, the radius of the air sphere.

Table 1. Huygens' parameters

		on the 1677 diagram	in the *Traité*
AP	Semi-major axis	105022	105032
AS	Semi-minor axis	93095	93410
AH	Section by interface	98473 (diagram) 98470 (words)	98779
AB	Normal deviation	100000	100000
AI	Vertical to tangent at B parallel to interface	99290	99324
BI	Interval from B to I	11898	11609
DB	Interval along line parallel to interface	86527	not used
DI	Interval from D to I	98425	not used
AK (Lμ – see note 16)	Radius of ordinary sphere in air	152678	156962
AX	Interval along interface for the undeviated ray	19941	17828

added a number to DB in order to generate DI, which makes sense only if all three of the intervals represented by that number, DB, and DI lie on the same line. Finally, and of greatest significance, the notebook contains two pages of computations immediately before the 'Eureka' (the first of the two includes an initial diagram; neither page is reproduced in the *Oeuvres*; see Fig. 4). And here we find BI given explicitly as 11898, and the normal deviation ($\angle$IAB) as $6°50'$.[17]

Since DI/AI must be the tangent of $\angle$PAI, and so of $\angle$HAS, we can deduce the interfacial angle that corresponds to Huygens' 1677 parameters using the procedure which he later specified in the *Traité*. And from that we can find the angle which is made with the crystal base by a vertical edge that forms a side of two obtuse facet angles. Next, the numbers for BI, AI entail the normal deviation.[18] And finally, the angle of incidence

[17] Codex 9, pg. 46a for the top figure, pg. 46b for the other. Note that Huygens' diagrammatic value for AH (98473 in Fig. 3) should be 98470 given the other three radii, and he did write 98470 in the remarks above the main figure.

[18] Huygens' procedure leads to the following expressions in terms of the interfacial angle ($\angle$i.f.a.):

$$\text{obtuse facet angle} = \angle\ \text{o.f.a} = \cos^{-1}[(\cot\{\angle\ \text{i.f.a}/2\}\cot\{\angle\ \text{i.f.a}\}]$$

$$\text{inclination of optic axis} = 180° - \tan^{-1}[\sin\{\angle\ \text{o.f.a}/2\}\ \tan\{\angle\ \text{i.f.a}\}].$$

The values for the obtuse facet angle and the inclination of the vertical edge, as well as for the interfacial angle, appear explicitly on pg. 430 in the *Oeuvres*, pg. 48 in Codex 9, immediately following the 'Eureka' page. In addition, the value for $\angle$IAP appears on the page before the 'Eureka' diagram which was omitted from the *Oeuvres*.

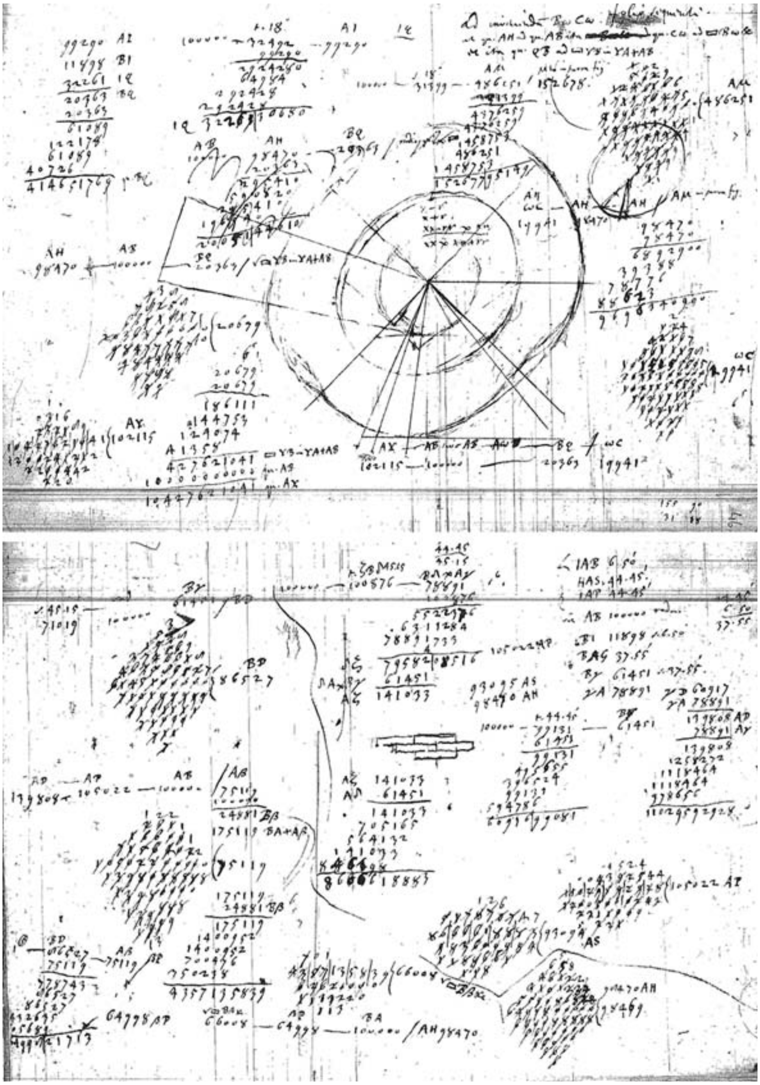

Fig. 4. Calculations from Huygens' notebook[19]

for the undeviated ray (AC) can be deduced from Huygens' value for AX combined with ∠IAP, the normal deviation (∠IAB), and the values for AP, AS. The number for AX must have been computed from some value for the angle of incidence, which we can accordingly find from AX itself using (1). Note that Bartholin had the undeviated ray parallel to the vertical edge just mentioned.

We see from Table 2 that by the summer of 1677 Huygens had produced his own values for the crystal angles and for the normal deviation, but that he continued to use

---

[19] These two pages are consecutive, but both are apparently numbered pg. 46 in the Codex.

Table 2. Bartholin's and Huygens' angles

	Bartholin in 1669	Huygens in 1677	Huygens in 1679
Inclination of optic axis to face	44°29′	44°45′	45°20′
Interfacial angle	103°40′	104°6′	105°
Obtuse facet angle	101°	101°18′	101°52′
Inclination of vertical edge	72°34′	72°	70°57′
Inclination of undeviated ray to crystal base	72°34′	72°	73°20′
Normal deviation	Unspecified	6°50′	6°40′

Bartholin's claim that the undeviated ray parallels the acutely-inclined vertical edge of the crystal. The diagram has another number – the radius of the sphere in air (Lμ) – that must have come from some measurement. It cannot result from the normal deviation, because that angle is used in conjunction with the reference AB and the crystal angle to set the ellipse radii. Neither can it come from a value for the ordinary index of refraction unless Huygens was prepared to assume at this point that the radius of the ordinary sphere in refraction is precisely the same as the semi-minor axis of the ellipsoid. But even if he were prepared to do so, the ordinary index would not likely have given him the number he wrote down. Though Huygens did not specify a value in 1677, Bartholin had given the ordinary index as 5 to 3. Using Bartholin's ratio combined with Huygens' 93095 for AS would set Lμ (hereafter denoted AK for reasons given above, note 16) to 155518, whereas Huygens' 1677 value is 152678. Where then might AK have come from?

One possibility is that Huygens obtained AK from the presumed incidence of 18° for the undeviated ray (in which case he would not have measured it by means of a refraction since he took the assumption that it paralleled a crystal edge from Bartholin). To do that he needed first to find AX from the angle of refraction, and then AK from the (equal) angle of incidence using (1), but, as noted above, there is no indication that he at this point possessed (2) and so its consequence (3).

There is however another, simpler route to AX from the refraction using the algebraic expression for the ellipse to generate the radius AC along which the refraction lies. AX then easily follows (Fig. 2):

$$
\begin{aligned}
AC &= \frac{(AS)(AP)}{\sqrt{AS^2\cos^2(\angle IAP - r) + AP^2\sin^2((\angle IAP - r))}} \\
AX &= AC(\sin(r) - \cos(r)\tan(\angle IAB))
\end{aligned}
\tag{4}
$$

And then AK follows as:

$$
AK = \frac{AH^2\sin(i)}{AX} \tag{5}
$$

Huygens' value of 19941 for AX in 1677 indeed follows from (4) (or for that matter from (3)) for an 18° angle of incidence/refraction, i.e for (as he then thought) a ray parallel to the acute vertical edge of the crystal. Huygens must therefore have produced one of (3) or (4) or their equivalents. Of the two, (4) is more easily found from the

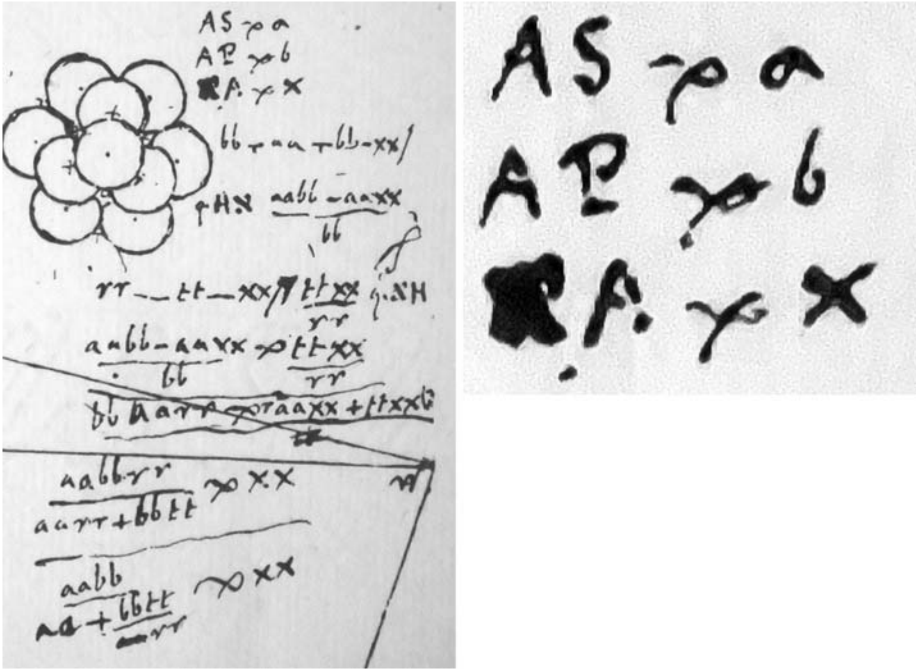

Fig. 5. Huygens' algebraic sketch

geometry of the ellipse and is accordingly the one that he most likely used. This is more than conjecture, because the sketch to the left of the main diagram (Fig. 5) shows an algebraic representation of the ellipse, with Huygens seeking an expression from it for what he labeled x. In the sketch he replaced AS, AP respectively with *a, b* and then introduced a pair *t,r* of variables, so that its final line reads:

$$\frac{AS^2AP^2}{AS^2 + \frac{AP^2t^2}{r^2}} \propto x^2 \tag{6}$$

The variable x represents some line that terminates in the end A of the semi-major axis, but Huygens has overwritten and obscured its other terminus. If we move in closer (rightmost image, Fig. 5) we see that he has drawn heavily over what might have been a C. In that case Huygens' x would be AC, and we can compare his (6) with our (4) by rewriting the latter as:

$$AC^2 = \frac{AS^2AP^2\left(\frac{1}{\cos^2(\angle IAP-r)}\right)}{AS^2 + AP^2\frac{\sin^2(\angle IAP-r)}{\cos^2(\angle IAP-r)}} \tag{7}$$

Note that $\angle IAP - r$ is the angle between the semi-major axis (AP) and the refraction; call it ϵ. The expressions $\sin^2 \varepsilon$, $\cos^2 \varepsilon$ would represent Huygens' *t,r* respectively, and it makes sense to express the relationships in terms of the angle between the ellipse's fixed axis AP and the refraction. Although the factor of AS^2AP^2 in (7) is obviously not constant, Huygens may have simply sketched a quick derivation before working out the details.[20] He needed the equivalent of (4) to calculate AX from an angle of refraction alone, and this algebraic fragment catches him in the act of deducing it.

Having obtained his value for AX from the undeviated ray, Huygens could then use (5) to calculate AK. However, this gives 150259, not the 152678 that he wrote on the diagram. Unless we assume that he erred to three significant digits out of six in a simple division and multiplication,[21] the discrepancy can mean only one thing: Huygens must have performed some sort of measurement in the summer of 1677. Is there any evidence as to what this might have been?

It's certainly possible that Huygens directly measured some refraction angle, generated a corresponding AX from it *via* (4) (or (3)), and then used (5) to compute AK. However, not only is there no evidence from 1677 concerning such a measurement, there are no refraction angles (using the natural crystal) anywhere else in Huygens' notebooks or in the *Traité* except for the undeviated ray,[22] which in 1677 Huygens had not even determined for himself. If there's no evidence that Huygens measured other refraction angles, then is there any evidence that he had a different method for generating AK from observation?

Indeed there is, and we can find it by comparing a fragment from the notebook with his published *Traité*. The fragment was likely produced as Huygens began his new investigations during the summer of 1679.[23] It reads:

> Refractio regularis paulo major ex ultima exactissima observatione quam 5 ad 3, et optime convenit ista 500 ad 293.
>
> Elevatio fundi minima major observata plurimis vicibus quam fiebat posita N 153456, et ad minimum quanta est ponendo N $\propto$ 158928
>
> Anglus radii transeuntis sine refractione fuit circiter 73 gr. Quantus ad minimum est ex observatione exacta.

---

[20] It's also possible, but not probable, that Huygens' x represents AX, in which case our equivalent would be rewritten for comparison as $AX^2 = \dfrac{AS^2AP^2\left(\frac{(\sin(r)-\cos(r)\tan(\angle IAB)}{\cos(\angle IAP-r)}\right)^2}{AS^2+AP^2\,\frac{\sin^2(\angle IAP-r)}{\cos^2(\angle IAP-r)}}$. Again the factor of AS^2AP^2 would not be constant.

[21] Which is altogether unlikely given the careful multiplications that we find in Fig. 4, where we also see that Huygens did not use logarithms, no doubt to keep his accuracy to 6 digits.

[22] And, of course, the ellipse-determining normal deviation.

[23] Dijksterhuis suggests the 1679 date on two grounds: first, because of its physical place in the bound notebook, and second, because of its (unspecified) 'content'. With respect to position in the notebook, which alone nearly guarantees a 1679 date, the fragment appears on pg. 92 *verso*, while the notebook's page 87 is headed (in Huygens' hand) 1679. With respect to content, the primary reasons for not placing it in 1677 are that Huygens suggests two values for AK which differ from the one on the 1677 diagram, and that the lettering for the several parameters is the same as in the *Traité*, which differs in this respect from the 1677 diagram. It nevertheless remains possible that Huygens had pursued further work along these lines in 1677, particularly since all of the parameters specified by the fragment, excepting only AK, have precisely the same values as in 1677, which differ from the final numbers in the published *Traité*. We do not know just when Huygens generated the *Traité*'s final values because the three extant documents from 1679 contain only one number, and it can't be used to compute any parameters at all (see below).

> Posita N 158928 et proportione refractionis regularis 500 ad 293 sit ratio N ad minorem spheroidis axem eadem quae N ad radium sphaerae extensionis luminis in refractione regulari.[24]

The *Oeuvres* prints another fragment immediately after the one just quoted:[25]

> CR, CG $\propto$ a $\propto$98473 fin. C. 6.50′ $\propto$ c $\propto$99290, sin 6.50′ $\propto$ LM $\propto$ d $\propto$11898. CM $\propto$ rad. 100000
> N ad CG proxime major quam 8 ad 5

In loose translation of the two passages:[26]

> [Codex 9, p. 90 *v*]
> CR, CG $\propto$ a $\propto$ 98473 fin. C. 6.50′ $\propto$ c $\propto$99290, sin 6.50′ $\propto$ LM $\propto$ d $\propto$11898. CM $\propto$ rad. 100000
> N to CG a bit greater than 8 to 5
> [Codex 9, p. 92 *v*]
> The regular refraction a little bit greater than [from] the last very exact observation [than = quam] 5 to 3, and it fits very well with [that (of) = ista] 500 to 293
> The smallest elevation of the [bottom = fundi] [is] greater [-] observed many times [-] than was the case taking N at 153456, and at the very least put[ting] N proportional to 158928.
> The [size of the] angle of the ray [that] from exact observation go[es] through without refraction was at the least about 73 gr.
> N having been taken at 158928 and the proportion of the regular refraction [having been taken] 500 to 293 [,] the ratio [of] N to the lesser axis of the spheroid is the same as that [of] N to the radius of the sphere of the extension of light in regular refraction.

According to the second sentence of the 92 *v* fragment, Huygens had "many times" observed the elevation of a point in extraordinary refraction, from which he concluded that N (i.e. AK) must be 158928 "at the very least". He had also performed a "very exact observation" of the ordinary index, finding it closest to 500/293. Immediately after this he tells us that the undeviated ray makes an angle with the horizontal of "at the least" 73°. These two measurements differ from the 152768 and 72° of 1677, whereas all of the other parameters on the preceding fragment of 90 *v* are unaltered. Note further that it is at this point that Huygens concluded the radius of the ordinary sphere must be the same as the semi-minor axis of the spheroid.

Sections 39 through 42 of the *Traité* reveal what Huygens was doing. There he describes, and then calculates, "a very singular effect", which is that the elevation of a point is different in extraordinary refraction depending upon which plane of incidence the eyes lie in, and that in all cases the depression of the image below the surface is greater than it is in ordinary refraction. In Sect. 41 he computes the effect for a principal section.

---

[24] *Oeuvres*, pg. 430.
[25] In the notebook this additional fragment appears on pg. 90 *verso*, whereas the 'Refractio . . .' passage follows it on pg. 92 *verso*.
[26] Huygens' Latin is loose at this point, so the translation, for which I thank Mac Pigman and Tony Grafton, can only be approximate.

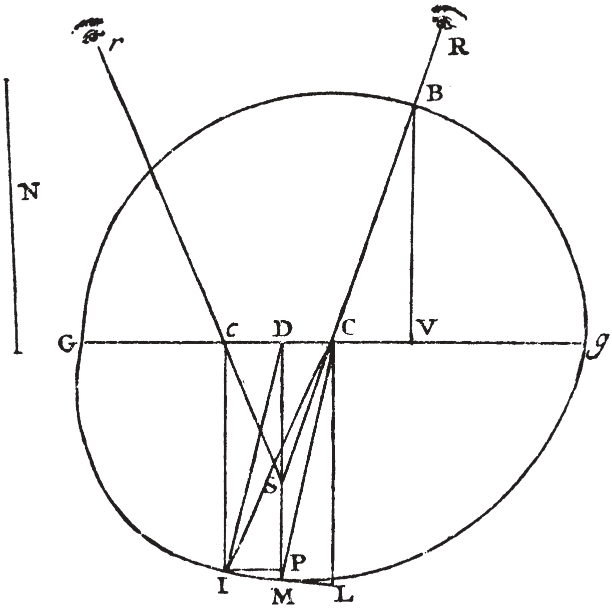

Fig. 6. Huygens' figure in the *Traité* for image depth observation

The method requires placing both eyes in the principal section (Fig. 6) in such a fashion that the extraordinary rays from a point I that reach the two eyes (*cr* and *CR*) emerge at equal angles from the crystal.[27] The extraordinary image of I will lie at S, where the backward-projections of the rays intersect; S lies on the perpendicular line through D, which bisects the distance between the points *c, C* of emergence. If the eyes are sufficiently far from the surface that BV can be taken for CB, then the apparent depression DS of Y will be very nearly a third proportional between AK and AH (in terms of the parameters in Fig 2):

$$\frac{AK}{DS} = \frac{DS}{AH} \tag{8}$$

Since AI represents the crystal height (Fig 2), (8) gives the ratio of the image's measured depression *d* to the height *h* of the crystal as

[27] Huygens did not explain how to do so in practice. However, the conditions will be satisfied, he noted, if I is so marked that ID is parallel to CM, where M lies on the ellipse directly below D, and CM is the normally-deviated ray from C to M. This might be done in the following way. First set a point M below the crystal, and then mark D directly above it on the crystal surface by observing its ordinary refraction. From D, and in the principal section, mark C where the extraordinary image of M emerges along the normal by computing, in a very good approximation, the distance CD as $h\tan(\delta)$, where h is the crystal height and δ is the normal deviation. Then, in a somewhat less accurate, but still good, approximation (which amounts to ignoring the distance ab in Fig. 2), mark I on the crystal bottom a distance CD from M. Set the eyes such that eye R sees I by extraordinary refraction in coincidence with the marked point C. The procedure requires considerable practice.

$$\frac{d}{h} = \frac{DS}{AI} \quad \text{and so} \quad AK = \left(\frac{d}{h}\right)^2 \left(\frac{AI^2}{AH}\right) \tag{9}$$

The combination of this result from the *Traité* with the fragment tells us how Huygens could obtain a value for AK without actually measuring any refraction angles at all. He took a straight rule, set it vertically on edge next to the crystal, and then marked on it the crystal's height and the apparent position of the raised point. He did this "many times", according to the fragment, and decided in the end that the best value for AK was 158928. It's not at all an easy observation to make, which is why Huygens had to do it "many times", and why as well that the fragment sets AK to "at the very least" 158928.

Although the 92*v* fragment almost certainly dates from 1679, the fact that it appears without any specification of the associated formula (8) may indicate that Huygens was already familiar with the method. If so, then we have a reasonable (indeed the only likely) source for his computation of AK in 1677. He would at that earlier date have measured a different value for the image height from the ones that he obtained two years later, which is hardly surprising considering his later remark that he had done the experiment "many times". Since it's quite clear that Huygens could not have used his 1677 value for the undeviated ray to compute AK, since there is no indication that he (ever) measured any other angle of refraction, and since we see that he did two years later refer to the method of image height for computing AK, it's plausible that this is what he did in 1677 as well.

Why would Huygens have worked with image height, which is at best an extremely difficult observation to make with any degree of accuracy? The answer is quite simple: it provided him with an independent, and uniquely specified, way to determine the critical parameter AK. Since Huygens always sought to produce the single best measurement that he could and never formed a resultant by combining multiple ones, he either had to choose a particular angle among all the possible ones to find AK, or else he had to find a different method that avoided the problem altogether, despite any other disadvantages. Because the method of image height requires a specific observational configuration, the problem of choosing an otherwise arbitrary angle of refraction evaporates. The notebook fragment 92*v* shows that Huygens was doing just this, certainly in 1679 and very likely in 1677 as well.

We return finally to the question of the undeviated ray, which Huygens had certainly not himself measured in 1677. The document dated July 3, 1679 describes a method for demonstrating that, contra-Bartholin (and his own earlier belief), the ray does not parallel an edge of the crystal. To prove the point Huygens marked where the ray should hit a point *B* below the crystal if it did parallel the edge, and then observed that *B*'s image did not emerge in coincidence with a point on the crystal's top marked along the edge-parallel from *B*. However, this measurement of the distance between the two points on the crystal top cannot yield the angle of the undeviated ray without also marking a point vertically below the emergence of the ray from *B*.

In fact, there's no feasible way at all to find the undeviated ray by means of a single operation. Huygens could set an incidence and then find the refraction, or he could set a refraction and then find the incidence. Or he could set a line at some angle and then find that the refraction from a point at its end does not lie along it (as in the July 3 measurement). But how to set points such that the incidence will equal the refraction? The difficulty is reflected in the very form of Huygens' (2), which forbids any reasonable method of solution in which the incidence would equal the refraction. Even if Huygens had AK from another experiment (as he surely did) there is no plausible method for deducing the undeviated ray.[28]

To find it Huygens must have worked essentially by trial and error. He already knew that the ray had to be nearly parallel to the crystal's acute vertical edge, so he may have begun his "exact observation" near there by setting a refraction and measuring the corresponding incidence. To check the result he would then have deduced the refraction from the measured incidence using his separately-obtained value of 158928 for AK. From the 1677 parameters – still in use in the later fragment – we find that a 73° incidence (to the horizontal) produces a $73°10'$ refraction. Little wonder that the fragment asserts the angle to be "at the least" 73°.

In the *Traité* Huygens used a value of 156962 for AK without specifying the experiment that led to it. He there also calculated that a $73°20'$ (horizontal) incidence produces, to the minute, a $73°20'$ refraction, This had suggested many years ago[29] that Huygens had computed the AK of the *Traité* by reversing the observation,[30] using either (4) or (3). It's now possible to be more specific. There are only two alternatives: either that he performed two different experiments, one for AK, and one for the undeviated ray, and that he just happened to get exactly the right value from the first to perfectly fit his observation of the second. This is, to say the least, unlikely, because the most accurate values for AK and for the undeviated ray are respectively 155350, $16°52'$ and not Huygens' 156962, $16°40'$. The second alternative is that Huygens performed the algebraic miracle, having measured AK, of deducing the undeviated ray from his geometry. Since neither is likely, it seems that Huygens did indeed compute AK from a measurement of the undeviated ray despite the absence of other probative evidence.

Why did Huygens not make his procedure explicit? He had not used the undeviated ray to find AK before, when he was trying to find out what its value might be. Moreover, in the *Traité* he did remark that he had found the ratio AK/AH to be "a little less than 8 to 5" from "observations of the irregular refraction" in a principal section. The notebook fragment writes the reverse, with AK/AG a bit greater than 8 to 5, but the comparison ratio remains the same, and there Huygens had found AK from image-height measurements. Of course, "a bit less than" yields no precise number, and Huygens certainly needed a good one for publication.

---

[28] To make the difficulty explicit, the undeviated ray satisfies the following unwieldy equation for AV in Fig. 2 (representing the normal deviation (∠IAP) by δ):

$$0 = \frac{AV^2}{AH^2} - \frac{\left(AI\sin\delta + (AH)(AV)/\sqrt{AK^2 - AV^2}\right)^2}{AI^2\cos^2\delta + (AI\sin\delta + (AH)(AV)/\sqrt{AK^2 - AV^2})^2}. \tag{10}$$

[29] Buchwald, J. Z. (1980). "Experimental investigations of double refraction from Huygens to Malus." *Archive for History of Exact Sciences* 21: 311–73.

[30] As Dijksterhuis reiterated (p. 179, note 56).

The audience for the *Traité* was not Huygens himself, as it had been in the notebook, but readers who needed convincing. This perhaps explains his presentation, in that Huygens may have decided to pack a persuasive result into the only computational example for an extraordinary refraction that he ever publicized. Neither did he actually mislead the reader, for he wrote only that the result follows given "the proportion of the refraction [*viz*. the parameters, including AK] being what we have just seen". It seems likely, accordingly, that Huygens intended only to show that his construction leads to the very existence of an undeviated ray (though there was the subsidiary aim to prove that such a ray did not parallel the crystal edge, and for that he might have been a bit more forthright). We see as well how extremely spare Huygens was in his measurements. Nowhere in manuscript or print is there the slightest evidence that he ever combined multiple observations to obtain a resultant value. He certainly did measure many times, but in the end Huygens always settled on the one number that he deemed the very best of all.

California Institute of Technology
1200 East California Blvd.
Pasadena, CA 91125
USA
buchwald@its.caltech.edu

(Received August 14, 2006)
Published online September 28, 2006 –