JED Z. BUCHWALD

AN ERROR WITHIN A MISTAKE?

AN ERROR WITHIN HELMHOLTZ'S ELECTRODYNAMICS?

In recent years much work has been done on the system of electrodynamics that was developed by the German polymath Hermann Helmholtz in the late 1860s and throughout the 1870s.[1] In 1874 Helmholtz used this new electrodynamics briefly to examine a situation that seems to be quite similar to one analyzed half a decade later by his student Heinrich Hertz in a manuscript written for Helmholtz's eyes. Though Hertz did not refer explicitly to Helmholtz's previous considerations of 1874, his analysis was based unequivocally on equations that were unique to Helmholtz. Yet in this particular application of the master's system, its creator in 1874 and his student in 1879 seem to have arrived at markedly different, indeed at conflicting, results.

Here, it seems, we have a situation in which one of the two must have erred either in calculation or else in setting out the problem's conditions. Both were using the very same system of electrodynamics, a system that was abandoned in Germany shortly after Hertz himself discovered electric waves late in 1887. We have accordingly found a most interesting circumstance, in which one of two practitioners apparently made some sort of mistake within the confines of a system that became altogether defunct about a decade later. It is as though we had come across a disagreement between, say, two proponents of 18th century affinity chemistry concerning a process that has no significance at all in the post-Lavoisieran world. We have stumbled across an error within a mistake: something done wrong within a now-rejected system.

Since the details of Helmholtz's electrodynamics have been given several times (see note 1), we may begin our excavation of error directly with the system's fundamental assumption, which is the existence of a 'potential' function from which the electrodynamic interaction between paired differential volumes of (e.g.) conducting bodies may be deduced. This function, P, can be interpreted, and used, as an energy of the system; it depends on the electric current within each volume element, as well as upon the distances between the elements. In its most general form P is:

$$P = \underbrace{-\frac{A^2}{2}\int_r\int_{r'}\frac{\mathbf{C}\cdot\mathbf{C}'}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'\mathrm{d}^3r}_{\text{`Neumann' function for extended objects}} \underbrace{-\frac{A^2}{2}(1-k)\int_r \mathbf{C}\cdot\left[\nabla_r\int_{r'}\mathbf{C}'\cdot\nabla_{r'}\left|\mathbf{r}-\mathbf{r}'\right|\mathrm{d}^3r'\right]\mathrm{d}^3r}_{\text{generalized term that vanishes if either circuit is closed}}$$

J.Z. Buchwald and A. Franklin (eds.), Wrong for the Right Reasons, 185–208.

Here A is a universal electrodynamic constant, $\mathbb{C}$ and $\mathbb{C}'$ are interacting currents, $\mathbf{r}$ runs from the origin to $\mathbb{C}$, $\mathbf{r}'$ runs from the origin to $\mathbb{C}'$, and k distinguishes among different permissible expressions for the interaction.

The function P was designed by Helmholtz to yield the by-then standard Ampère force between current-bearing circuits when the circuits are closed. Indeed, if either of the interacting systems to which the volume elements d^3r and d^3r' respectively belong is closed, then this most general expression for the potential reduces to its first term. That expression (for closed, linear circuits) was first obtained by Franz Neumann at Königsberg in his successful attempt to find a single function from which both electrodynamic force and electromagnetic induction could be obtained by, respectively, space and time differentiation.[2] In the system considered by Hertz in 1879, as well as in the actual experimental systems that were examined by Helmholtz and his collaborators earlier in the 1870s, one of the current-bearing objects always forms an effectively closed circuit, so that, with them, we may limit our considerations here to the implications of this first term in P.

The forces that act on the current bearing objects are calculated by varying the function P. The volume elements may experience two kinds of effect. Each may be acted on by a force that tends to move the element physically from one location to another in the usual way (i.e., by producing an acceleration equal to the force divided by the mass of the element) – contemporary language referred to this kind of force as *ponderomotive*. Each may also experience an action that tends to change the current that exists within it – or, in common parlance, an *electromotive force* (or *emf*). The important point for our purposes is this: Helmholtz's potential function yields the same *emf* as other theories of the day (and as modern electrodynamics) *only* when the element that is being acted on itself forms part of a closed system; otherwise Helmholtz's scheme entails an altogether novel force. The second major task that Helmholtz assigned his apprentice, the talented young Heinrich Hertz, was to see whether one could polarize dielectrics by means of electromagnetic induction. To do so Hertz thought to use this new *emf* that his mentor's system entailed; to that end he produced a feasibility study for Helmholtz's own eyes, a *prospectus* as it were of what might be done (Hertz, 1879).

Yet in this work of 1879 that Hertz drew up for Helmholtz himself, and wherein he used his mentor's novel electromotive force, he arrived by computation at results that seem to conflict with ones that Helmholtz had explicitly set out in print five years before. Hertz had surely read Helmholtz's paper, though he might have overlooked the remark. But even if he had missed it, or if he had not read the paper at all, how, using Helmholtz's own electrodynamics, could he have reached a different result? Is it a simple case of a mistaken calculation by a young apprentice? Or had Helmholtz himself erred? And, if so, why did neither Helmholtz (who read Hertz's MS) nor Hertz apparently ever notice it? Have we just found an error within a mistaken theory? Or is there something more to it?

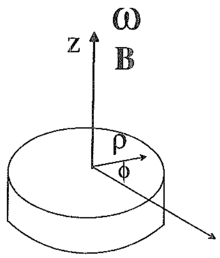

Figure 1. Helmholtz's spinning disk.

HERTZ AND HELMHOLTZ SEEM TO DISAGREE

The problem first appeared in 1874. In April of that year Helmholtz ended an article entitled "Kritisches zur Elektrodynamik", which considered objections to his formulation of electrodynamics, by pointing to a specific case in which his system yielded results that are different from those that are implied by all of the others. Helmholtz described the situation in the following words:

> Imagine a metal disk [see Figure 1] that is spinning rapidly about its axis and that is crossed by magnetic lines of forces that are parallel to the axis and symmetrically distributed about it, then the edge of the disk will be electrified according to the Ampère law,[3] but it will not be according to the potential law. (Helmholtz, 1874, p. 762)[4]

To understand what was at stake here, let's begin with modern electrodynamics and generalize the problem to an object that moves with a velocity **v** through a magnetic field **B**; a point in the object is specified by the vector **r** that is drawn to the point from the origin of coordinates, and the object's center of mass is itself located by the vector $\mathbf{r}_{cm}$ (see Figure 2). The object will experience an electromotive force (*emf*) **F** at a given point that is given by the cross-product there of the velocity with the magnetic field:

$$\mathbf{F}_{MAX} = \mathbf{v} \times \mathbf{B} \tag{1}$$

(*The "Ampère" expression for the electromotive force*)

If our object spins with angular velocity ω about its center of mass then the linear velocity **v** at the point in the object that is specified by the vector **r** will be:

$$\mathbf{v} = \omega \times (\mathbf{r} - \mathbf{r}_{cm}) \tag{2}$$

(*The velocity at a point of a spinning object*)

Consequently, according to the Ampère expression the electromotive force at **r** will be:

$$\mathbf{F}_{MAX} = [\omega \times (\mathbf{r} - \mathbf{r}_{cm})] \times \mathbf{B} \tag{3}$$

(*The Ampère emf for a spinning object*)

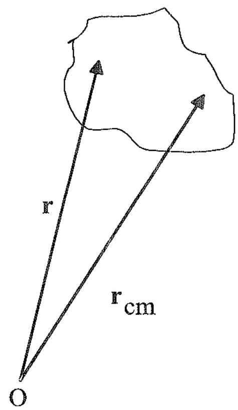

Figure 2. Object vectors.

Suppose now that the angular velocity is parallel to the magnetic field. According to this equation, the *emf* will not vanish.

Yet according to Helmholtz in 1874 the *emf will* disappear in these circumstances. Why? The answer is, in one sense, quite simple: Helmholtz's expression for the *emf* contains an additional term that can in the right circumstances annul the Ampère expression. According to Helmholtz, the electromotive force that acts on an object which moves with velocity $\mathbf{v}$ in the presence of what we shall call a vector potential $\mathbf{A}$ is:

$$\mathbf{F}_{\text{HELM}} = \underbrace{\mathbf{v} \times (\nabla \times \mathbf{A})}_{\text{Ampère term}} - \underbrace{\nabla(\mathbf{v} \cdot \mathbf{A})}_{\text{Helmholtz term}} \tag{4}$$

(*emf according to Helmholtz*)

As the Appendix below shows, this auxiliary vector is defined by Helmholtz (and so by Hertz) primitively in terms of currents (or derivatively in terms of magnetization). For reasons that will become clear below, it's important to note that this potential gains significance altogether from its role as the vector that a current multiplies in calculating the energy of the system comprised of the current in question and the currents or magnetization with which it is interacting (*vide* equation (21)).[5] In Helmholtz's energy-based electrodynamics the vector potential has no other function or meaning than this, but, just because of its immediate presence in the energy, the potential was more fundamental than the forces to which the energy gave rise.

The new term in Helmholtz's expression for the force can cancel out Ampère *emf*s – and, according to Helmholtz, it does so when an object spins about an axis that is parallel to a magnetic field which is symmetric about the axis. Despite the fact that the young Hertz used exactly Helmholtz's formula for this very force, he – unlike Helmholtz himself in 1874 – did not find that the force must vanish when the angular velocity and the magnetic field are parallel to one another, as we shall see in detail.

Yet in both cases essentially the same magnetic field and velocity are involved. One of the two, it seems, must have erred in computation.

HERTZ'S SPHERE

Hertz did not consider precisely the same configuration that his mentor Helmholtz had, but the one which he did examine provides a more general case that embraces Helmholtz's, as we shall see. Hertz's particular goal was to find a way of experimentally testing whether or not the electromotive force that is generated by motion through a magnetic field can polarize dielectrics just as it can generate currents in conductors. To do so he thought to use the force that would be generated in a small object by spinning it in the earth's magnetic field. To that end he had first to calculate the magnetic force at the earth's surface, for which he used auxiliary functions that were in reasonably standard German employ at the time.

Hertz began with a quantity $\boldsymbol{\lambda}$ which he used to represent the potential of the earth's magnetization $\mathbf{M}$ (i.e., its magnetic moment per unit volume). With λ given by $\int (\mathbf{M}(r')/|\mathbf{r}-\mathbf{r}'|)\,\mathrm{d}^3r'$, the corresponding vector $\mathbf{A}$ that is to be used in Helmholtz's equation (4) has the form $-\nabla \times \boldsymbol{\lambda}$:[6]

$$\mathbf{A} = -\nabla \times \boldsymbol{\lambda}$$

where

$$\boldsymbol{\lambda} \equiv \int (\mathbf{M}(r')/|\mathbf{r}-\mathbf{r}'|)\,\mathrm{d}^3r' \tag{5}$$

(*The vector potential A for magnetization M*[7])

We can substitute $\mathbf{A}$ into Helmholtz's basic expression for the force to obtain:[8]

$$\mathbf{F}_{\mathrm{HELM}}^{\mathrm{MAG}} = \mathbf{v} \times \nabla(\nabla \cdot \boldsymbol{\lambda}) - \nabla(\mathbf{v} \cdot (\nabla \times \boldsymbol{\lambda})) \tag{6}$$

(*The force expressed in terms of the magnetization potential*)

In equation (6) the force is labeled $\mathbf{F}_{\mathrm{HELM}}^{\mathrm{MAG}}$ to emphasize that it is not as yet in a form appropriate to Hertz's specific application to the earth, in that $\boldsymbol{\lambda}$ may to this point derive from any source of magnetization whatsoever.

Let's now follow Hertz's application of the formula to the case of an object spinning in the earth's field, adding in a few details that he omitted in order to facilitate our comparison of his results with those of Helmholtz. Take the earth's magnetization m to be directed along the z axis, and assume that the magnetic effects which are responsible for the earth's field are localized near the earth's center, so that we can take the distance from the earth's center to our spinning object also to be the effective distance between the object and the earth's magnetization (see Figure 3). As a result, the vector λ will be:

$$\boldsymbol{\lambda}_{\mathrm{HTZ}} = \frac{m}{r}\mathbf{e}_z \tag{7}$$

(*Hertz's magnetization potential $\boldsymbol{\lambda}_{HTZ}$ for the earth's field*)

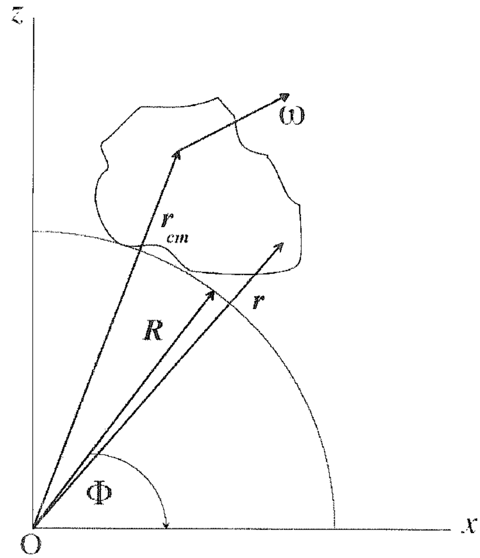

Figure 3. Spinning object near earth's surface.

Hertz did not provide the vector **A** for calculating the magnetic force since he had no specific need for it given expression (6), but for future reference we note that **A** has the following exact and approximate forms:

$$\mathbf{A} \text{ is exactly } -\nabla \times \left(\frac{m}{r}\mathbf{e}_z\right) \text{ or approximately } \frac{m}{R^2}\cos(\Phi)\mathbf{e}_y \tag{8}$$

Here $\mathbf{e}_z$ lies along the polar axis, R is the earth's radius, Φ is the latitude, and $\mathbf{e}_y$ is orthogonal to a meridian plane.

Hertz next moved almost directly to give expressions for the *emf*s that result according to equations (6) and (7) when a small object spins near the surface of the earth. First of all, let's assume, as he did, that our object's angular velocity lies entirely in the plane formed by the earth's polar axis (to which the magnetization is assumed to be parallel) and the line from the earth's center to the object. That is, our object spins only about an axis that lies in the plane of the local meridian – we will not examine the effect of an east–west component. Since the earth's field is axially symmetric about the polar (say z) axis, we can in full generality consider the forces that act in any plane section that contains the axis. For simplicity we will take the xz plane as the one in which we calculate forces.[9] The object's angular velocity accordingly has (by assumption) no component along the y axis, but it may have components along the z (polar) and x (equatorial) axes. Denote the latitude at our object by Φ, and assume as well that the object's dimensions are small to first order in respect to the radius R of the earth. In the final result we can accordingly replace r (the distance to the object point) with R (the radius of the earth – see Figure 3).

We can now proceed to substitute Hertz's magnetization potential (equation (7)) into Helmholtz's force (equation (6)), after which we replace both of the distances r and r_{cm} with the earth's radius R (Figure 3). We then drop all expressions in which the third or higher power of the earth's radius appears in the denominator, on the grounds that other terms remain that contain a factor of only $1/R^2$, as we shall see in a moment. This last assumption completely removes the expression that corresponds to the Ampère *emf* (*viz.* $\mathbf{v} \times \nabla(\nabla \cdot \boldsymbol{\lambda})$).[10] Limiting our consideration to the xz plane, we find with Hertz that the spinning body will experience the following *emf*:

$$\begin{aligned} F^x_{\mathrm{HTZ}} &= -\frac{m}{2R^2}\omega_z \cos(\Phi) \\ F^y_{\mathrm{HTZ}} &= 0 \\ F^z_{\mathrm{HTZ}} &= -\frac{m}{2R^2}\omega_x \cos(\Phi) \end{aligned} \tag{9}$$

(*The emf on the spinning object according to Hertz*)

These *emf*s vanish altogether at the poles and are a maximum at the equator, for a given angular velocity.

We ask next what direction the magnetic force itself has at the latitude Φ. For consistency we must use Hertz's expression for the magnetization potential in our computation (equation (7)). Since the corresponding magnetic force must be $-\nabla \times (\nabla \times \boldsymbol{\lambda})$, we find (again under the approximation that in the end we replace both r and r_{cm} with R):

$$\begin{aligned} B^x_{\mathrm{HTZ}} &= \frac{m}{R^3}(3\sin(\Phi)\cos(\Phi)) \\ B^y_{\mathrm{HTZ}} &= 0 \\ B^z_{\mathrm{HTZ}} &= \frac{m}{R^3}(2 - 3\cos^2(\Phi)) \end{aligned} \tag{10}$$

(*The magnetic force corresponding to Hertz's magnetization potential*)

We can obviously adjust the angular velocity so that it parallels the magnetic force at a given latitude.[11] In fact, we can rewrite Hertz's expressions for the *emf* in terms of the local components of the magnetic force in the following way:

$$\begin{aligned} F^x_{\mathrm{HTZ}} &= -\frac{R}{2}\omega_z(B^z_{\mathrm{HTZ}}\cos(\Phi) - B^x_{\mathrm{HTZ}}\sin(\Phi)) \\ F^y_{\mathrm{HTZ}} &= 0 \\ F^z_{\mathrm{HTZ}} &= -\frac{R}{2}\omega_x(B^z_{\mathrm{HTZ}}\cos(\Phi) - B^x_{\mathrm{HTZ}}\sin(\Phi)) \end{aligned}$$

To take a simple example, we can locate ourselves at the equator, where the magnetic force runs along a north–south axis (c.f. equation (10), with Φ set to zero), and where the *emf* reaches a maximum. We can set a sphere of radius r_s, say, spinning about its center around this same axis (tangent to the local meridian), in which case the *emf* will point directly downwards (see Figure 4). In this same situation, the

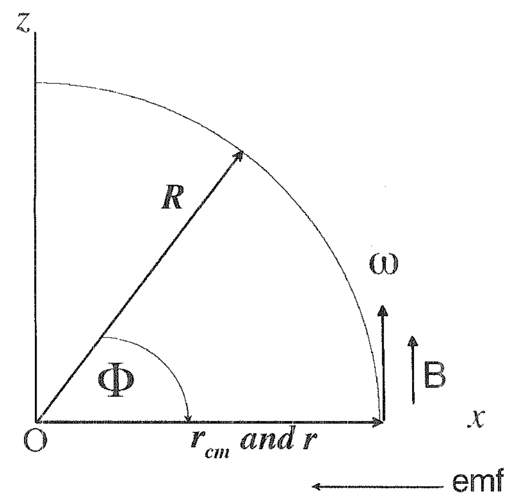

Figure 4. The *emf* at the equator for an object spinning parallel to the polar axis, according to Hertz's equations.

Ampère expression ($\mathbf{v} \times \mathbf{B}$) yields an *emf* directed at each point along a line that is perpendicular to the object's axis of spin, aiming directly away from the axis and towards the surface of the sphere. However, the Ampère *emf* will be incomparably smaller than this new one that Hertz has calculated, since it will contain a factor r_S, whereas the Hertz force contains a corresponding factor R.[12] That is, the new force is larger than the Ampère *emf* by the immense ratio R/r_S. Of course, the Hertz force should not exist at all according to Helmholtz's remarks in 1874, who had at the time used precisely the same expression for calculating the *emf* that his student Hertz used later (*viz.* equation (4)). Turn now to Helmholtz's claim.

HELMHOLTZ'S DISK

Helmholtz had not specifically discussed an object of any shape whatsoever spinning in an arbitrary direction in the earth's magnetic field. His comment referred to a disk that turns about its axis of symmetry in a field of magnetic force that is parallel to, and symmetric about, the axis. Under these conditions, Helmholtz had asserted in 1874, the Ampère expression requires the existence of an *emf* that is directed from the central axis towards the disk's perimeter. But his own force law, he continued, implies that there will be no *emf* at all. We will turn below to the reasoning that may lie behind Helmholtz's claim. Let's first consider whether, and if so in what manner, it applies to the situation that Hertz envisioned half a decade later.[13]

One might argue that the two situations (Helmholtz's and Hertz's) differ from one another because Helmholtz specified a magnetic field that is symmetric about and parallel to a disk's axis of spin, whereas Hertz considered the earth's field, which certainly seems not to satisfy the requirement, *vide* equation (10). However, Hertz's spinning object is vastly smaller than the earth, and in its vicinity the earth's field should certainly be effectively uniform, thereby trivially fulfilling Helmholtz's symmetry requirement. Nevertheless, Helmholtz's conclusion implicated the symmetry of a field of magnetic force, whereas Hertz's calculation was based upon a specific expression for the magnetization potential, from which the force was computed. In order to clarify the plausible assertion that the (locally insignificant) inhomogeneity of Hertz's magnetic force cannot be the source of the difference between his and Helmholtz's claims, we will first connect Hertz's calculation to a vector **A** that does yield a strictly uniform force.

We need to find a vector whose curl will be equal to a homogeneous magnetic field **B**.[14] One such is $\mathbf{A}_\mathrm{r}$:

$$\mathbf{A}_\mathrm{r} = \tfrac{1}{2}(\mathbf{B} \times \mathbf{r}) \tag{11}$$

(*A vector potential that produces a uniform magnetic field*)

Inserting $\mathbf{A}_\mathrm{r}$ into the general Helmholtz expression for the force ($\mathbf{F}_{\mathrm{HELM}}$, equation (4)) we obtain (naming the result $\mathbf{F}_\mathrm{r}$):

$$\mathbf{F}_\mathrm{r} = \tfrac{1}{2}\mathbf{v} \times \mathbf{B} + \tfrac{1}{2}\boldsymbol{\omega} \times (\mathbf{B} \times \mathbf{r}) \tag{12}$$

(*First expression for the emf corresponding to A_r*)

This new *emf* can be written in strict mathematical equivalence as:

$$\mathbf{F}_\mathrm{r} = \tfrac{1}{2}(\mathbf{r} - \mathbf{r}_{\mathrm{cm}}) \times (\mathbf{B} \times \boldsymbol{\omega}) + \tfrac{1}{2}\boldsymbol{\omega} \times (\mathbf{B} \times \mathbf{r}_{\mathrm{cm}}) \tag{13}$$

(*Second expression for the emf corresponding to A_r*)

We next insert into equation (13) the very same expressions for the magnetic field that result from Hertz's magnetization (equation (10)). In addition, we also approximate the center-of-mass distance (r_{cm}) by the earth's radius (R). Doing so yields, as expected, precisely the same expression for the *emf* on the spinning object that Hertz himself had obtained (equation (9)). In other words, the vector $\mathbf{A}_\mathrm{r}$ produces the very same *emf* as the vector **A** (equation (8)) that corresponds to Hertz's magnetization potential, λ_{HTZ} (equation (7)), when the same approximations are used.

Since we now see that Hertz's expressions for the *emf* follow perfectly well from a calculation based on the assumption that the local magnetic force is uniform in direction and magnitude, it follows that the difference between his and Helmholtz's assertions can have nothing to do with any slight local inhomogeneity. If Hertz's claim is correct, then it seems that Helmholtz's simply cannot be, and *vice versa*.

Or have we missed something essential here? To see whether or not we have, turn first to Helmholtz's original statement. Helmholtz had there referred explicitly to "magnetic lines of force ... that are parallel to the axis and symmetrically distributed about it". Although he used the phrase "magnetic lines of force", Helmholtz just might have been thinking of a field of vector potential, since his entire discussion of the *emfs* involved in motion proceeds from his fundamental interaction energy, which is formulated in terms of the vector potential and not (by necessity) the corresponding magnetic force. If that were so, then Helmholtz's conclusion would be almost obvious, given the foundation of his electrodynamics in variational calculations based on interaction energy: for if the vector potential is itself symmetric about the disk's axis of rotation, then the potential that will be seen by any point of the rotating sphere or disk must always be the same – in which case the energy-variation that underpins Helmholtz's calculations can yield no resultant force at all, just as he asserted.[15] Under this interpretation there is no conflict between Hertz's and Helmholtz's claims; we are instead left with a sloppy statement on the part of Helmholtz – and worse, one that would not correspond to any reasonable experimental situation, since the originating currents follow the vector potential in direction.[16]

There is another possibility. What if Hertz's *emfs* are not the only ones that are consistent with the assumption that the field of magnetic force is uniform (or symmetric about the axis of spin)? This seems unlikely, since we have already found that there is nothing at all wrong with his computation, and, moreover, that it is entirely compatible with the local uniformity of the magnetic force. But to imply a proposition is not necessarily to be implied by it.

Let's return to the vector potential that corresponds to a uniform magnetic force. We considered $\mathbf{A}_r$, which, we saw above, produces Hertz's *emf* when we require (as we may) that the expressions for the $\mathbf{B}$ field in the resultant force (equation (12)) be the same as those that are implied by Hertz's magnetization. But this is not the only vector potential that can produce the requisite magnetic force. In fact, we can clearly add any constant, or the gradient of any function, to $\mathbf{A}_r$ and still obtain what we need if we are concerned only with the resultant magnetic force. For example, we could if we like replace the distance $\mathbf{r}$ to the point in the object at which the *emf* is calculated with the distance from the object's center of mass to that point, because the additional term that results (namely $-\frac{1}{2}(\mathbf{B} \times \mathbf{r}_{\mathrm{cm}})$) is itself a constant. The potential $\mathbf{A}_{\mathrm{cm}}$ would then be:

$$\mathbf{A}_{\mathrm{cm}} = \tfrac{1}{2}[\mathbf{B} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}})] \tag{14}$$

(*A magnetically-equivalent vector potential*)

Note that if the magnetic field $\mathbf{B}$ is parallel to the angular velocity ω then this vector potential will itself parallel the linear velocity $\mathbf{v}$. Note also that our new vector potential is axially symmetric since its direction and magnitude always have the same values with respect to the disk's radius. This is not true for $\mathbf{A}_r$ because the position vector $\mathbf{r}$ is not perpendicular to the disk's axis (*vide* note 15).

If we now insert $\mathbf{A}_{\mathrm{cm}}$ into Helmholtz's formula then we obtain, after considerable but standard manipulation (recalling that the location of the center of mass and the angular velocity of the spinning body are both to be considered constant):

$$\mathbf{F}_{\mathrm{cm}} = \mathbf{v} \times (\nabla \times \mathbf{A}_{\mathrm{cm}}) - \nabla(\mathbf{v} \cdot \mathbf{A}_{\mathrm{cm}}) = \tfrac{1}{2}\left[\mathbf{v} \times \mathbf{B} - \omega \times (\mathbf{B} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}}))\right] \tag{15}$$

(*First expression for the emf corresponding to A_{cm} according to Helmholtz's formula*)

We can immediately see that this new force differs by the term $-\frac{1}{2}[\mathbf{v} \times \mathbf{B} + \boldsymbol{\omega} \times (\mathbf{B} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}}))]$ from the expression (equation (3)) for the Ampère *emf*. Of course, the force that derives from $\mathbf{A}_{\mathrm{r}}$ (equation (12)) also differs from the Ampère expression. The question is whether our new force, which derives from $\mathbf{A}_{\mathrm{cm}}$, differs in an appropriate manner from the one that is implied by $\mathbf{A}_{\mathrm{r}}$.

Indeed it does. The new expression can be manipulated to yield, in strict equivalence:

$$\mathbf{F}_{\mathrm{cm}} = \tfrac{1}{2}\left[(\mathbf{r} - \mathbf{r}_{\mathrm{cm}}) \times (\mathbf{B} \times \boldsymbol{\omega})\right] \tag{16}$$

(*Second expression for the force corresponding to A_{cm} according to Helmholtz's formula*)

According to this equivalent second expression, the *emf* will indeed vanish altogether whenever the angular velocity parallels the magnetic field. We have therefore found a vector potential that yields a uniform field of magnetic force and that nevertheless produces the very effect that Helmholtz had claimed.

How can this be so? The answer is deceptively simple: although a constant addition to the vector potential has no affect at all on the magnetic force, it certainly may have one on the electromotive force according to Helmholtz's electrodynamics, because Helmholtz's general expression for *emf* contains the additional term (in comparison to Ampère) $-\nabla(\mathbf{v} \cdot \mathbf{A})$. Even if the addition (call it $\mathbf{A}'$) to the vector potential is constant, this extra term in the force will yield two novel contributions: namely, $-(\mathbf{A}' \cdot \nabla)\mathbf{v}$ and $-\mathbf{A}' \times (\nabla \times \mathbf{v})$. Neither of these necessarily vanishes, because $\mathbf{v}$ may depend upon r (*vide* equation (2)). As a result, $\mathbf{F}_{\mathrm{cm}}$, but not $\mathbf{F}_{\mathrm{r}}$, does indeed disappear when the angular velocity is parallel to the magnetic force. It's instructive to rewrite the force that arises from the Hertz potential ($\mathbf{A}_{\mathrm{r}}$) in the following manner, since we can then easily see how it differs from the one that arises from $\mathbf{A}_{\mathrm{cm}}$:

$$\mathbf{F}_{\mathrm{r}} = \overbrace{\underbrace{\tfrac{1}{2}\left[(\mathbf{r} - \mathbf{r}_{\mathrm{cm}}) \times (\mathbf{B} \times \boldsymbol{\omega})\right]}_{\mathbf{F}_{\mathrm{cm}},\ \text{the Helmholtz force from } \mathbf{A}_{\mathrm{cm}}} + \underbrace{\tfrac{1}{2}\left[\boldsymbol{\omega} \times (\mathbf{B} \times \mathbf{r}_{\mathrm{cm}})\right]}_{\text{addition from } \mathbf{A}_{\mathrm{r}}}}^{\text{The Hertz force}} \tag{17}$$

(*Comparison of the Hertz and Helmholtz forces*)

Here we see clearly that the Hertz force can yield a result even when the Helmholtz *emf* vanishes altogether. Unlike field theory, Helmholtz's system is manifestly not gauge-invariant, and in this case of the spinning disk or sphere we have found a situation in which the lack of invariance has a testable consequence.

We can naturally ask whether Helmholtz might have envisioned such an expression as $\mathbf{A}_{\mathrm{cm}}$. If we recognize that he, unlike Hertz (who started from the earth's magnetization), began with a field of magnetic force and a spinning object, then it seems plausible that Helmholtz would have thought of this expression for the vector potential, had he produced any at all, and not the one that Hertz's lengthy computation entailed. Unlike Hertz, who naturally reckoned from the earth's center, Helmholtz (thinking of a locally-produced magnetic field) would no doubt have worked in terms of local cylindrical coordinates, placing the origin at the center of his spinning disk. The potential $\mathbf{A}_{\mathrm{cm}}$, unlike $\mathbf{A}_{\mathrm{r}}$, contains the vector $\mathbf{r} - \mathbf{r}_{\mathrm{cm}}$, or ρ, which represents the distance from the center of mass of the spinning object to the point on it at which we wish to calculate the *emf*. This same distance appears in the velocity $\mathbf{v}$ (equation (2)) of such a point. Accordingly, if Helmholtz had wondered at all about an appropriate vector potential to correspond to his magnetic field, then he would likely have used the very same vector that appears in the velocity, thereby ensuring the absence of *emf*. We will turn in a moment to the possible course of Helmholtz's reasoning during the year following the publication of his remark concerning the *emf* in a spinning disk, but let's first consider the difference between Hertz's and Helmholtz's attitudes in respect to this sort of problem.

Hertz was in an altogether different frame of mind from Helmholtz when he considered the spinning sphere. Helmholtz in 1874 was looking for a situation that contrasted strikingly with the claims of the Ampère *emf*. Hertz was looking for a way to test whether electromagnetic induction can polarize dielectrics. Where Helmholtz was looking to provide evidence for a new force law, Hertz was looking to use this same force law as a tool in order to see whether a particular kind of novel effect that would otherwise be difficult to produce could actually be elicited. Hertz therefore began directly with the specific physical situation that he had in mind, and he then proceeded in a straightforward way to calculate the force from it. He started with what he took to be the most fundamental assumption possible, namely that the earth's field results from a magnetic dipole located near its center. Hertz had to work with the dipole's potential, and not the force to which it gives rise, because Helmholtz's law was expressed in terms of potentials.

Do we have any evidence concerning Helmholtz's own thoughts in respect to the requirements of his new force law, based as it was on the vector potential and not on magnetic force? To answer that question, let's first return to Helmholtz's original statement of 1874. There Helmholtz specified a magnetic field that is symmetric about the disk's axis of rotation; he said nothing about the vector potential *per se*, or even about the sources of the field. We saw above that we can produce a trivially symmetric magnetic field – i.e., a constant one – using either of the following two vector potentials (with $\mathbf{B}$ constant of course):

$$\mathbf{A}_{\mathrm{r}} = \tfrac{1}{2}(\mathbf{B} \times \mathbf{r}) \quad \text{or} \quad \mathbf{A}_{\mathrm{cm}} = \tfrac{1}{2}[\mathbf{B} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}})]$$

Neither of these two potentials corresponds to a physically-realizable distribution of (closed) currents, simply because the curl of their curl – which represents current – vanishes.[17] Nevertheless, the difference between these two expressions contains a hint that may be historically significant.

We have seen that $\mathbf{A}_{\mathrm{r}}$ yields a force on a spinning disk or sphere when the magnetic field parallels the rotation, whereas $\mathbf{A}_{\mathrm{cm}}$ does not. If the field is parallel to the angular velocity, then we can rewrite $\mathbf{A}_{\mathrm{cm}}$ as $\frac{1}{2}[(\frac{B}{\omega})\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}})]$. The expression $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{cm}})$ is just the linear velocity $\mathbf{v}$ at the circumference of our rotating sphere or disk. Here, then, the vector potential circulates symmetrically about the disk's axis, while the corresponding magnetic field parallels the axis.[18]

Consider any given radius of the rotating disk. No matter what the position of the radius may be at any given moment, it always sees precisely the same value $\mathbf{A}_{\mathrm{cm}}$ because it has always the same velocity $\mathbf{v}$. And here we perhaps spy a clue to Helmholtz's reasoning during the year after his remark was printed. Suppose we assume that the vector potential is produced by currents that are concentric to, and symmetric about, the disk's axis. In such a case as well, the rotating radius will always see the same potential. It is not a difficult leap from the symmetry of an $\mathbf{A}_{\mathrm{cm}}$ that produces a constant magnetic field to the potential (call it $\mathbf{A}_{\mathrm{symcurr}}$) that is produced by axially-symmetric currents proper. Neither $\mathbf{A}_{\mathrm{cm}}$ nor $\mathbf{A}_{\mathrm{symcurr}}$ will produce any *emf* in the rotating disk or sphere, and for precisely the same reason.

We may now fruitfully examine the consequences of the fact that the *emf* will vanish only when the vector potential is axially symmetric. Specifically, suppose that $\mathbf{A}$ has the general, axially-symmetric form $h(\rho)\mathbf{e}_\varphi$ where $h(\rho)$ depends solely on the distance from the central axis. Here the cylindrical-coordinate ρ specifies the distance to a given point from the z axis, while φ specifies the angle of ρ in a plane orthogonal to z. Then the magnetic field $\mathbf{B}$ becomes $[(h + \rho h')/\rho]\mathbf{e}_z$, and $\mathbf{F}_{\mathrm{emf}}$ vanishes.[19] This magnetic field is itself axially symmetric (although orthogonal to its vector potential), so we have now found a situation that corresponds directly to Helmholtz's requirement and claim in 1874. The point that Helmholtz seems to have missed is this: namely, that $\mathbf{A}$ fields which are not themselves axially-symmetric can nevertheless generate $\mathbf{B}$ fields that are, with non-zero *emf*s resulting thereby. One such $\mathbf{A}$ field, for example, is $h(\rho)\mathbf{e}_\varphi + \varphi\mathbf{e}_\rho$. The corresponding magnetic field is then $[(h + \rho h')/\rho - 1]\mathbf{e}_z$, which is itself axially-symmetric, but the *emf* no longer vanishes, becoming in fact $-\omega\mathbf{e}_\rho$.

Hertz's magnetization potential for the earth is just another example, albeit one in which the magnetic field is symmetric about the spin axis by virtue of its near uniformity in the neighborhood. This is most simply understood by considering the potential's approximate form, in which we replace the vectors to the object point and to the object's center of mass with the earth's radius. For then we can at once see that the approximate potential (see equation (8)) has the form $\mathbf{B} \times \mathbf{r}$ (see equation (10)), and this, as we have seen, does not abolish the Helmholtz *emf*.

DISAGREEMENT AVOIDED, WITH REMARKS ON MISTAKES, NOVELTY AND PRACTICAL WORK

We began our discussion by pointing to a conflict between Helmholtz and his student Hertz concerning the *emf* that is generated in a spinning object subject to a magnetic field. It's certainly possible that the difference remained unresolved, and that it was perhaps never even recognized at all by either of them. But Helmholtz did not cease working on electrodynamics after the paper containing his claim about the spinning disk was printed. Not at all – he continued to write articles on the subject, and a good deal of related experimental work occurred in his Berlin laboratory. Is there any evidence in this subsequent activity that Helmholtz ever recognized, if only implicitly, that his remarks concerning the spinning disk were problematic?

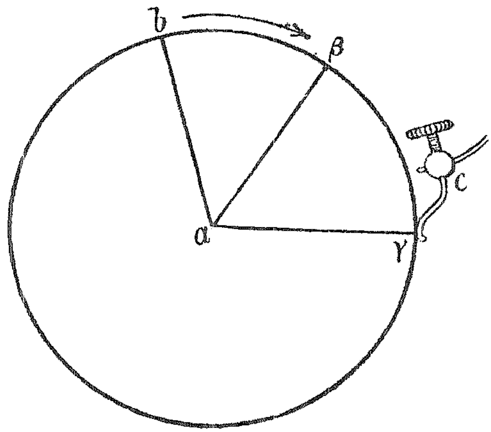

Figure 5. Helmholtz's test for *emf.*

Indeed there is. The very next year Helmholtz made the following remark in a paper concerning experiments done on induction produced by motion in open circuits. Helmholtz wrote:

> Let the endpoint a of the conductor (ab Fig. 1 [Figure 5 here]) be fixed, b however being able to rotate in a circle about a, further let the acting magnets and current elements be so arranged that the first of these constitute rotationally-symmetric bodies, whose magnetic axes, as well as whose axes of rotational symmetry, coincide with the normal erected at the midpoint of the circle, while the circuits build concentric circles about this axis. With such an arrangement, the relative position of the radius $a\beta$ with respect to the magnets or the currents is precisely the same as {that of} ab; the electrodynamic potential has the same value in both cases, namely zero, and the potential law would as a result have the consequence that in this case no electromotive force will act along it during the course of the rotation of the radius ab into position $a\beta$. (Helmholtz, 1875, p. 782)

Here we see that Helmholtz has now recognized the conditions that must be satisfied in order to guarantee the absence of *emf.* The currents that act upon the moving radius all lie in concentric circles having as axis the line about which the arm *ab* in the figure rotates. Further, any magnetic bodies have their axes of magnetization and rotational symmetry along this same axis, and so here too the rotating radius can never see any change in the vector potential.[20] Not only are all magnetic fields axially-symmetric, so too are the corresponding vector potentials.

Clearly, during the time between his remark the year before and this one Helmholtz had understood the need to specify conditions on the symmetry of the vector potential rather than the magnetic field. He was undoubtedly pressed to do so by the demands of an experiment to test the *emf* produced by motion, for that required producing an appropriate physical configuration of currents and magnets. The situation described here is what Helmholtz had had in mind the previous year, but with a notable difference: the magnetic field in this new situation is not necessarily parallel to the arm's axis of rotation. It is however always axially-symmetric, as are any currents. For the latter reason alone there can be no resultant *emf*.

And so we have solved our apparent conundrum. There is in the end no persistent disagreement between Helmholtz and his student Hertz, because Helmholtz in 1875 altered his inadequate remark of 1874. Hertz's quick and easy use of Helmholtz's equation for *emf* needs no explanation at all, at least insofar as a putative conflict with Helmholtz is concerned. Unlike Helmholtz himself, who had deduced the expression, Hertz had learned it. For him applying the formula to an object spinning in the earth's field was just an exercise in using what he had learned from Helmholtz at a comparatively early stage in his career. For Helmholtz, on the other hand, the new expression for *emf* had come as the result of considerable work trying to build a general foundation for electrodynamics. He had not learned it as a student, and for Helmholtz, its creator, the new formula undoubtedly did not have the character of intuitive directness that, for some time in the early 1880s, it had for the young Hertz.

Four different moments in the production of a novel physical system are nicely illustrated here. Within five years of the system's initial production by Helmholtz we find him applying it incorrectly on paper. Helmholtz had indeed erred within the confines of his own new system. But not for long: the very next year, faced with the concrete demands of a real experimental structure, Helmholtz corrected his error. Then, four years later, the neophyte Hertz, who had learned the new system without having been thoroughly immersed in alternatives to it, applied the scheme almost mechanically, without questioning the elements in it that experiments had begun to make problematic even to its originator.

Novelty, error, error rectified, and finally rote application – these are issues that raise questions for understanding how systems that live both on paper and in the world of material devices evolve. Initially, the specific novelties of Helmholtz's system had little relevance for the contemporary electrodynamics laboratory; there simply weren't any devices that worked with the new forces that Helmholtz had created on paper. Neither were any experimental oddities clarified thereby. More to the point, the world of electrodynamic devices and objects had long been designed and understood on the basis of symmetries that were scarcely compatible with the new system.

Symmetries often constitute critical elements in apparatus design, usually for intensely practical reasons. In lens design, for example, it had always been important to avoid astigmatism, which meant that the lens had to have the same form in any plane section through its center and including its lenticular axis. Moreover, the very manner in which lenses were ground meant that any asymmetries that did occur had to be the result of undesirable and hard to control factors. Lenticular symmetry accordingly represented both abstract desiderata (the avoidance of astigmatism) and practical necessity (lens grinding methods). The motors and induction coils of electrodynamic devices – the existing world that Helmholtz's system had to accommodate – had similar design and practical symmetries built in. Apparatus builders and paper analysts had long concentrated on the magnetic forces that push motors around and that induce electric currents. It wasn't practical to make, or to calculate, complicated force patterns, and so devices were constructed with extremely simple symmetries. Usually the goal was to keep the magnetic field as uniform as possible within the motor or the induction coil, and to avoid complexity in the winding of coils or armatures. Symmetry of calculation connected to symmetry of design.

Uniform magnetic forces, or forces that are nicely and simply distributed about well-chosen axes, were all that were needed to build working apparatus until Helmholtz's intervention in 1870. Intuitions had been developed over decades for designing and building apparatus that had the right kinds of symmetries to produce the desired actions. Helmholtz himself undoubtedly possessed just this kind of intuitive sense, and it was precisely this that led him into error in 1874, because his new system broke apart the prevailing concordance of symmetries.

It's not generally wise to attribute 'error' to work done long ago, because it is entirely too easy to ignore contemporary factors that make reasonable what was done or said at the time, or to import into past work irrelevant present views. But error *per se* certainly can and does exist. It can be recognized at the time, and, even if it isn't, the historian who has mastered the tools and techniques of the era has license to point out mistaken calculations or claims that shed revealing light on what took place. It is not easy to specify a precise set of rules that might govern the act of error-excavation (and in itself error is not perhaps intrinsically interesting), but at least this much should hold true: the error-excavator must be reasonably certain that the error-maker could have been persuaded to acknowledge and to correct his mistakes had they ever been pointed out.

In Helmholtz's case there is no doubt that he would have acknowledged error, because he in fact did so (albeit implicitly) the very next year by altering his specification for a symmetry that would abolish the electromotive forces. During the year between 1874 and 1875 Helmholtz recognized that intuitions based on force symmetries did not work for his scheme. Symmetries had rather to apply to the vector potential than to the magnetic force that arises by taking its curl, which meant that apparatus had to be designed by arranging the wires themselves according to the desired patterns. Intuitions about symmetric forces had to be replaced by intuitions about symmetrically-placed wires.

This is particularly significant when we recognize that Helmholtz's system worked entirely and directly with entities that formed the tangible electrodynamic workplace. His scheme did not base itself upon electric particles, as did his rival Wilhelm Weber's (and many others in Germany at the time), nor did it work at a fundamental level with force fields, as the British did. It spoke instead of wires that carried currents, or of electrically-polarized dielectrics, or of magnetic bodies. These were its primordial elements, at least in the 1870s, and despite the certain fact that Helmholtz did think that something more fundamental might lie hidden beneath the tangible world, he did not for the most part build his theories at a deep level on conjectures about the invisible realm.

Although our subject has not been a large one since it does not involve many people over a long period of time, it does have broader implications than might seem to be the case because its argument runs counter to contemporary historical trends that resolutely deny the existence of anything beyond the purely local – of beliefs and behaviors that transcend immediate circumstances and that may hold across national, cultural, and economic boundaries. Much history of science today sees all events as irremediably local, as having no counterparts among other people, at other times, and in different places. Innumerable articles have been written in recent times with the adjective 'local' prominently displayed for admiration in title or body to show that the author does not adhere to the disreputable notions of unity or generality. Heterogeneity in society is, no doubt, morally and socially salutary. The last century provides too many examples of what happens when passions for homogeneity govern life and desire. But, to state what should be trivially obvious, attempts to achieve internal consistency and general applicability in technical systems do not necessarily have much in common with attempts to impose social uniformity by tyrants or fanatics.

The events that we have examined are certainly 'local' in the sense that they took place in particular places, at specific times, and among certain people. And they are local even in their express content, since the peculiarities of Helmholtz's electrodynamics were pursued mostly in Berlin. But in a broader sense much is entirely general here. Helmholtz erred in thinking that it was sufficient to specify a symmetry for the magnetic force, and he later knew as much. Hertz correctly and even mechanically carried out an internally-consistent computation based on Helmholtz's system. It is entirely reasonable to assert that Hertz in 1879, but not Helmholtz in 1874, worked correctly and without error. Locality in our case pertains rather to the specifics of Helmholtz's system, which were certainly not shared by many of his German or British contemporaries, than to the pragmatics of calculation or even of instrumentation. It would have been quite possible for the Maxwellian J.J. Thomson, e.g., to uncover Helmholtz's 1874 error, to follow Hertz's 1879 calculation, and to see exactly how Helmholtz's system had correctly to be applied.

Moreover, Helmholtz fully intended that his electrodynamics should be uniquely correct – that all others would have to fall in some way or other under its sway or else be abandoned altogether. In this he was no different from any of his contemporaries, who however held different views as to the best system to adapt, or for that matter from any mathematically or experimentally oriented investigator since antiquity. Views can be nuanced, and often have been, concerning such things as whether a particular scheme is physically as well as mathematically significant, or even whether mathematics can be used at all. But I do not know of anyone who has ever maintained that two systems or computations, each of which claims to treat essentially the same physical domain in similar ways, and which have conflicting empirical consequences, can both be correct.

APPENDIX: HELMHOLTZ'S POTENTIAL AND CORRESPONDING FORCES

The close links between Helmholtz's electrodynamics and energy considerations have been discussed several times by historians, as have his deductions of the corresponding electromagnetic forces.[21] Nevertheless, it is worthwhile reproducing as closely as possible Helmholtz's own analyses in order to capture the full flavor of his theory in the manner that he intended. We will rely on previous historical work, including my own, but will diverge from it in presentation and detail in order to adhere closely to Helmholtz.

Helmholtz's own theory of electrodynamics was presented in a series of 11 papers published from 1870 through 1881. Two among these developed the system in elaborate detail, specifically 1870b and 1874. The 1870 paper developed the consequences of Helmholtz's generalized electrodynamic potential, in particular (as its title suggests) for currents in conductors at rest, but also (and importantly) for dielectrics. Here Helmholtz was not concerned with either the mechanical force that acts to move current-bearing bodies, or the electromotive force engendered by changes in the configuration of systems in which they exist. In response to a series of intense criticisms by, among others, Wilhelm Weber, Eduard Riecke and Carl Neumann in Germany, and Joseph Bertrand in France, Helmholtz carefully worked out the forces implied by his theory.

Although part of Helmholtz's purpose was to consider the most general possible form for a potential function that would be compatible with the generally accepted laws that govern closed circuits, we will here limit our considerations to that part of the potential which is given by a generalization of the expression developed by Franz Neumann. This expression was originally developed solely for linear, closed circuits. One of Helmholtz's major assumptions was that the elements in the Neumann integral could be considered independently, thereby extending the expression to open circuits. In addition, Helmholtz examined three-dimensional currents, to which we will here limit our own considerations.[22]

We begin with the electrodynamic 'potential' that two three-dimensional current distributions establish when one at least of them forms a closed system:[23]

$$P = -\frac{A^2}{2}\int_r\int_{r'}\frac{\mathbf{C}\cdot\mathbf{C}'}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3r'\mathrm{d}^3r \tag{18}$$

In this expression for P, the integrations both occur over all space, which counts each pair of volume elements $\mathrm{d}^3r'\mathrm{d}^3r$ twice. If, as Helmholtz remarks (Helmholtz, 1874, p. 732) the currents occur in physically separated conductors, and the integrations each occur over only one set, then the factor of $\frac{1}{2}$ may be dropped. In addition, the currents $\mathbf{C}$, $\mathbf{C}'$ are *fluxes*: that is, they represent the quantity of charge per unit time per unit area that flows in a given direction. Helmholtz's generalization to three-dimensions of the procedure established originally by Franz Neumann then yields forces according to the following rules.

The ponderomotive force – the force that moves a body physically – is (following Helmholtz's sign convention) to be found from the negative gradient of this function, with the operator affecting only the locus of the body on which the force acts. Helmholtz accordingly set the negative variation of the potential function equal to the product of the force sought by the variation in position of the object. If the loci of points in the body carrying current $\mathbf{C}$ are specified by vectors $\mathbf{r}$, then the force $\mathbf{F}_{\mathrm{pmf}}$ that would act on an element d^3r as a result of its change in position from $\mathbf{r}$ to $\mathbf{r}+\delta\mathbf{r}$

must accordingly satisfy variational equation (19):

$$\int (\mathbf{F}_{\mathrm{pmf}} \cdot \delta\mathbf{r})\, \mathrm{d}^3 r + \delta_{\mathrm{r}} P = 0 \tag{19}$$

The subscript 'r' in δ_{r} indicates that the displacement of the object is completely arbitrary. Note that $\mathbf{F}_{\mathrm{pmf}}$ depends only upon the configuration of the system and the magnitudes of the currents.

Other forces, called *electromotive* (or *emf*), may also exist that act to change the magnitudes of the currents themselves. With Helmholtz we consider the *emf* that would act on a unit current in a given direction. Such a force is given by the positive rate of change with time of the potential, with the proviso that the *emf* must not depend upon the amount of charge per unit time (that is, the *linear* current) which flows through the object being acted upon. In the case of three-dimensional currents, we construct an appropriate variational equation purely formally by taking the scalar product of the *emf* with whatever current flux $\mathbf{C}$ exists at its locus, and then setting the result equal to the time-rate of change of the potential function. We will subsequently impose the condition that the resulting *emf* must be independent of linear current. This gives equation (20):[24]

$$\left(\int_r \mathbf{F}_{\mathrm{emf}} \cdot \mathbf{C}\, \mathrm{d}^3 r \right) \delta t - \delta_t P = 0 \tag{20}$$

The subscript 't' in δ_t indicates that the change in the potential is calculated over an arbitrary increment of time. As we will see, the variation in the position of the object during this time interval is not arbitrary: it is determined by the object's velocity, and the corresponding variation represents the change as seen by the moving object.

To facilitate computation Helmholtz in 1870 had introduced a vector $\mathbf{U}$, which allowed him to express the potential in a manner that provided in the end a compact representation of the forces:[25]

$$\mathbf{U}(\mathbf{r}) = \int_{r'} \frac{\mathbf{C}'(r')}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}^3 r' \rightarrow P = -\frac{A^2}{2} \int_r \mathbf{C} \cdot \mathbf{U}\, \mathrm{d}^3 r \tag{21}$$

The constant A in (18) and (21) is fundamental in Helmholtz's electrodynamics, but our discussion here does not depend upon it, and so it has been suppressed below for notational simplicity. Note that in this form the energy P depends directly on the properties of what we shall now call the vector potential $\mathbf{U}$. Since everything in Helmholtz's electrodynamics follows from the basic energy expressions, intuitions about how to set up exemplary problems must be developed about current and potential, and not about the resulting forces, since the forces are derivative, not fundamental, quantities.

Helmholtz worked as follows.[26] Begin with the general expression for $\mathrm{d}P$, which contains both $\mathbf{C}$ and $\mathbf{C}'$. Consider an infinitesimal portion of current-bearing material, the volume $\mathrm{d}^3 r$ of the element being $(\mathrm{d}\sigma)(\mathrm{d}r)$. Choose $\mathrm{d}\mathbf{r}$ such that it is parallel to the current flux $\mathbf{C}$ in our element. We may then write the product $\mathbf{C}\, \mathrm{d}^3 r$ in the equivalent form $(C\, \mathrm{d}\sigma)\, \mathrm{d}\mathbf{r}$. As a result, the contribution $\mathrm{d}P$ that this element will make to the entire potential will be:[27]

$$\mathrm{d}P = -(C\,\mathrm{d}\sigma)\,\mathbf{dr} \cdot \mathbf{U} \tag{22}$$

Since the variation is done without any consideration of the linear current $C(\mathrm{d}\sigma)$, we can now set this product to one and ignore it altogether.

Return to equation (20), and consider the contribution to the entire variation $\delta_t P$ that comes from the circuit element $\mathbf{C}\,\mathrm{d}^3r$, which must now be set to **dr** in the variation for the *emf* as well:

$$\delta_t(\mathrm{d}P) = (\delta t)\mathbf{F}_{\mathrm{emf}} \cdot \mathbf{dr} \tag{23}$$

From equation (22) we can calculate the variation of the element $\mathrm{d}P$ in terms of **U**:

$$\delta_t(\mathrm{d}P) = -\delta_t(\mathbf{U} \cdot \mathbf{dr}) \tag{24}$$

Consequently we have:

$$(\delta t)\mathbf{F}_{\mathrm{emf}} \cdot \mathbf{dr} = -\delta_t(\mathbf{U} \cdot \mathbf{dr}) \tag{25}$$

Helmholtz had now to compute the change that arises when the affected object moves in relation to the external currents, and when the external currents are themselves allowed to change *in situ*, with the virtual displacement of the object occurring as a result solely of its motion with a velocity **v** during an infinitesimal time. A modern procedure can be used greatly to simplify the computation, but it is historically instructive explicitly to follow Helmholtz's own route.[28]

Let's consider separately the two parts into which the variation divides. The first part represents the change in **U** that is seen by a point fixed in the element when the element moves from a place where **U** has one value to a place where its value is different, together with the temporal change in **U**. The second part of the variation represents the change in the value of $\mathbf{U} \cdot \mathbf{dr}$ that occurs as a result of the alteration in the element's length. Hereafter italic boldface ($\boldsymbol{U}$) represents a vector as seen by a point that is fixed in the element:

$$\delta_t(\mathbf{U} \cdot \mathbf{dr}) = \underbrace{(\delta_t \boldsymbol{U}) \cdot \mathbf{dr}}_{1} + \underbrace{\mathbf{U} \cdot \delta_t(\mathbf{dr})}_{2} \tag{26}$$

Here $\boldsymbol{\delta}_t(\mathbf{dr})$ is the change in the length of the element that occurs as a result of its motion. Since δ_t and d commute,[29] we may replace $\delta_t(\mathrm{dr})$ with $\mathrm{d}(\delta_t \mathbf{r})$. And since the differential operator d is itself $(\mathbf{dr}) \cdot \nabla$, the second part of the variation may be written:

$$\underbrace{\mathbf{U} \cdot \delta_t(\mathbf{dr})}_{2} = \mathbf{U} \cdot [(\mathbf{dr} \cdot \nabla)\delta_t \mathbf{r}]$$

Furthermore, $\delta_t \mathbf{r}$ itself is just the virtual change in **r** produced by motion with velocity **v** during the time interval δt, i.e., $\mathbf{v}\delta t$:

$$\underbrace{\mathbf{U} \cdot \delta_t(\mathbf{dr})}_{2} = \mathbf{U} \cdot [(\mathbf{dr} \cdot \nabla)\mathbf{v}]\,\delta t \tag{27}$$

As for the first part of the variation in (26), we want to express our result in terms of the value of U at a fixed point in space – not at a fixed point of the displaced element. However, the U that appears in (26) refers to a specific point in the moved element. $\delta_t U$ must therefore be calculated using the material derivative (following the point) in order to express our results in terms of U at a fixed spatial point:

$$\delta_t U = \left[\left(\frac{\partial \mathbf{U}}{\partial t} + \{(\mathbf{v} \cdot \nabla)\mathbf{U}\}\right)\delta t\right] \tag{28}$$

Combining equations (25) through (28) yields (after dropping the common scalar factor δt):

$$\mathbf{F}_{\mathrm{emf}} \cdot \mathbf{dr} = -\left[\left(\frac{\partial \mathbf{U}}{\partial t} + \{(\mathbf{v} \cdot \nabla)\mathbf{U}\}\right)\right] \cdot \mathbf{dr} - \mathbf{U} \cdot [(\mathbf{dr} \cdot \nabla)\mathbf{v}] \tag{29}$$

After manipulation, the right-hand side of equation (29) can be put in a form that contains the scalar product of a vector with **dr**. Equating that vector to $\mathbf{F}_{\mathrm{emf}}$ yields Helmholtz's expression for the electromotive force:

$$\mathbf{F}_{\mathrm{emf}} = -\frac{\partial \mathbf{U}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{U}) - \nabla(\mathbf{v} \cdot \mathbf{U}) \tag{30}$$

California Institute of Technology, USA

NOTES

[1] Buchwald (1985, 1993a, 1993b, 1994), Darrigol (1993a, 1993b, 2000), Kaiser (1993).
[2] Archibald (1989), and Olesko (1991, chap. 5). See also Darrigol (2000).
[3] This is the electromotive force due to motion that follows from an electrodynamics that also yields Ampère's original bodily force between circuit-elements carrying electric currents. In what follows we will for the sake of brevity refer to this as the "Ampère" *emf,* although Ampère himself certainly never obtained any such thing since he did not of course discover electromagnetic induction (though he probably did observe it: see, e.g., Hofmann, 1995, chap. 8).
[4] The original reads: "Denken wir uns eine drehende Metalscheibe, schnell um ihre Axe rotirend, und von magnetischen Kraftlinien durchzogen, die der Axe parallel, und rings um die Axe symmetrisch vertheilt sind, so wird der Rand der Scheibe nach dem Ampère'schen Gesetze elektrisch werden, nach dem Potentialgesetze nicht."
[5] In the case of currents $\mathbf{A}(\mathbf{r})$ has the form $\int (\mathbf{C}(\mathbf{r}')/\,|\mathbf{r}-\mathbf{r}'|)\, d^3r'$; in the case of a magnetization $\mathbf{M}$ the vector $\mathbf{A}$ becomes $-\nabla \times \int (\mathbf{M}(r')/\,|\mathbf{r}-\mathbf{r}'|)\, d^3r'$. See below, note 7.
[6] The vector $\boldsymbol{\lambda}$ functions in a way that is analogous to that of the scalar potential for electric charge since $\nabla^2\boldsymbol{\lambda} = -4\pi\mathbf{M}$. By the time that Hertz arrived in Berlin, methods for calculating the force exerted by magnetic distributions were well known, although specific details might differ from author to author. The route from Helmholtz's definition of the auxiliary vector **A** in terms of currents (see equation (21) in the Appendix, where the vector **U** stands for **A**) to the specification of **A** for magnetization was also well known, though again details would differ from author to author. Helmholtz in any case provided the details that Hertz would have needed in this respect, if he did not already know them, in Helmholtz (1870, pp. 617–119).
[7] Expression (5) for the vector potential due to magnetization is nowadays rather unfamiliar. Using the Coulomb gauge we today write (ignoring a sign difference due to Helmholtz's convention) $\int ((\nabla_{r'} \times \mathbf{M}(r'))/\,|\mathbf{r}-\mathbf{r}'|)\, d^3r'$. The two forms are however equivalent: they differ

by a term $\int (\nabla_{r'} \times (\mathbf{M}(r')/\,|\mathbf{r} - \mathbf{r}'|))\, d^3r'$, and this vanishes on integration over all space if the magnetization is localized. See Jackson (1975), sec. 5.8.

[8] Hertz actually worked from Helmholtz's expression for the force (equation (4)) modified by the introduction of an auxiliary scalar function χ equal to $-\nabla \cdot \boldsymbol{\lambda}$, which facilitated the comparison of Helmholtz's expression for the *emf* with one that had been derived in 1864 on the basis of Weber's electrodynamics by Emil Jochmann (Jochmann, 1864). Assuming that $\nabla^2 \boldsymbol{\lambda}$ vanishes – which simply means that the force calculation holds for points that are located outside the magnetization proper – then Hertz could replace $\nabla \times (\nabla \times \boldsymbol{\lambda})$ with $-\nabla\chi$ in the expression $-\mathbf{v} \times (\nabla \times (\nabla \times \boldsymbol{\lambda}))$ for the Ampère term.

[9] Precisely because Hertz computed the force assuming a magnetic dipole located at the earth's center his coordinate system had its origin there as well. We will see in what follows that the choice of coordinate systems is closely connected to the apparent difference between Hertz and Helmholtz.

[10] It removes as well an extremely small term that is linear in the distance from the object's center of mass to the point in it at which the *emf* is to be computed.

[11] The angular velocity will parallel the magnetic force if its equatorial and polar components are in the ratio $3\sin(\Phi)\cos(\Phi)/(2 - 3\cos^2(\Phi))$.

[12] For this particular example, the Ampère *emf* would be $r_s\omega_z B_z$, whereas the Hertz *emf* would be $-(R/2)\omega_z B_z$.

[13] Helmholtz's spinning disk corresponds to a slice of Hertz's sphere taken orthogonally to the sphere's axis of rotation. Whatever consequences correctly hold for Helmholtz's disk will *ipso facto* hold as well for Hertz's sphere by treating the sphere as the limit of a series of stacked disks.

[14] Note again that a field uniform in direction and magnitude is trivially symmetric about its direction and so is clearly a special case of Helmholtz's requirement.

[15] We can easily understand this by remarking that $\mathbf{A}_r$ (equation (11)) implicates the distance $\mathbf{r}$, which implies that the potential seen by a point on the rotating arm must depend upon its angular position since $\mathbf{r}$ does not remain the same during the rotation.

[16] In such a situation the curl of the vector potential (i.e., the magnetic force) would always be tangent to concentric circles having a central axis as their common normal, and it could vary with distance from the origin along, and in the plane normal to, this common central axis. The magnetic field would accordingly circulate about the disk's axis, and to produce this would require something like a closed solenoid that coils around the disk's perimeter.

[17] If, that is, we consider them to be exact and not just approximations that are useful for nearly homogeneous magnetic fields.

[18] Certainly the magnetic field is also (trivially, because constant) symmetric about the axis, but we have already seen that this alone will not guarantee the absence of *emf* (since $\mathbf{A}_r$ also produces $\mathbf{B}$): in addition, the originating vector potential must circulate symmetrically.

[19] For h equal to $B\rho$, with B constant, this reduces to $\mathbf{A}_{cm}$(equation (14)).

[20] This claim in respect to axes of magnetization is not altogether obvious, though Helmholtz certainly recognized it (perhaps as an implication of the possibility of replacing magnetization with closed, bounding currents). It can be demonstrated, as follows. Consider an object spinning with angular velocity ω about the z axis and with its center of rotation located along that axis at a distance h from the origin. At the origin place a magnetic dipole whose axis also lies along z. The velocity of an arbitrary point r in the object, and the vector potential at that point will be:

$$\mathbf{v} = \omega\mathbf{e}_z \times (x\mathbf{e}_x + y\mathbf{e}_y)$$
$$\mathbf{A} = \nabla \times \frac{\mathbf{e}_z}{r}$$

From these we easily discover:

$$\mathbf{A} = \frac{\mathbf{v}}{r^3}$$

And the corresponding 'Helmholtz' force becomes:

$$\mathbf{F} = \mathbf{v} \times (\nabla \times \mathbf{A}) - \nabla(\mathbf{v} \cdot \mathbf{A}) = \mathbf{v} \times \left(\nabla \times \frac{\mathbf{v}}{r^3}\right) - \nabla\left(\frac{v^2}{r^3}\right)$$

We thereby find that $\mathbf{v} \times (\nabla \times \mathbf{A})$ and $\nabla(\mathbf{v} \cdot \mathbf{A})$ are equal to one another, which reduces the force to zero.

[21] Buchwald (1985, 1994) and Darrigol (1993a, 2000).

[22] Buchwald (1994, pp. 25–27) for the forces that arise among linear circuits.

[23] Helmholtz (1874, p. 717) gives the potential for linear circuits, and extends it to three-dimensional ones on pp. 730–731. Helmholtz (1870, p. 568) gives the general expression for the first time.

[24] Helmholtz (1874, p. 744).

[25] Helmholtz (1870, p. 568).

[26] Helmholtz (1874, pp. 742–745).

[27] Note that the factor of $1/2$ disappears on taking the differential. The factor emerges in the first place because otherwise the contribution to the potential from $\mathbf{C} \cdot \mathbf{C}' d^3 r d^3 r'$ would be counted twice, assuming both integrals to extend over all space. In taking the differential, however, the integration over r is dropped, and the factor of $1/2$ consequently vanishes. Formally, the factor disappears on taking the differential because the total potential, P, is symmetric in the product $\mathbf{C} \cdot \mathbf{C}'$.

[28] See Darrigol (2000, Appendix 5), which indicates that Helmholtz final's result can be obtained by calculating the convective derivative of the vector potential $\mathbf{U}$ under the requirement that the integral of the potential around a curve remains constant under a virtual displacement, i.e., that $\delta_t \oint \mathbf{U} \cdot d\mathbf{r}$ must vanish. Helmholtz reasoned entirely in terms of a differential element by considering explicitly both the change in the value of a vector that is seen by a point fixed in the element, and the change in the element's length. He did not examine the value of a curve-integral during a deformation.

[29] Because the variation of a differential element of length is equal to the difference between the variations of its endpoints.

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